OPEN EXTENSIONS OF FUNCTIONS

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CERTIFICATE

The thesis, "Open Extensions of Functions" submitted by Mr. Jogendra Kumar Kohli is original research developed and written by him. In my estimation it satisfies the requirements for the degree of Doctor of Philosophy in Mathematics. It has not been nor will it be submitted to any other institution for such a degree or for any award.

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SYNOPSIS

Present thesis is a study of various techniques of extending a (continuous) function to an open (continuous) function and the applications of these techniques. During the last two decades there has been some interest in trying to improve the behavior of maps by extending their domains. For example, in 1953 Whuburn defined a unified space of mapping and showed that every continuous map is the restriction of a continuous compact map. Similarly Krolevec proved in 1967 that each locally perfect map can be extended to a perfect map. More recently, in 1969, Dickman obtained the same result for arbitrary maps. Motivated by the work of Whyburn, Krolevec and Dickman one is led to ask the following question : Is every continuous function from a topological space into another is the restriction of an open continuous function ? This question is answered in the affirmative by giving three different proofs and many applications of the techniques developed in the process are obtained.

Chapter one contains preliminaries and background results that are needed throughout the thesis. In chapter two it is shown that a (continuous) function f from a topological space X into a topological space Y can be extended to an open (continuous) function f^* from X^* onto Y. Here the space X^* is obtained as a quotient of a disjoint topological sum of X and copies of Y. Also the topological properties of X, Y and mapping properties of f are related with the topological properties of X^*.

Chapter three discusses some applications and results of techniques of chapter two. It has been shown how the results and techniques of chapter two can be utilized to obtain analogues and improvements of various results on

open mappings. As an illustration of this fact improvements or analogues of recent theorems of Arhangelskiǐ, Čoban and Proizvolov on finite-to-one mappings are obtained. Similarly improvements or modifications of recent theorems on open mappings and dimension by Hodel, Keesling and Nagami are obtained.

Chapter four deals with two other methods of extending a continuous function to an open continuous functinn. It follows from the first method of construction that every continuous function is the restriction of a projection map. The same method of extension also implies that any continuous function from a topological group (respectively, topological vector space) into another is the restriction of an open continuous homomorphism (respectively linear mapping). In the end we give a method of unifynng domain and range of a continuous function so as to yield a meaningful open extension. A modification of the last method is useful in obtaining partial improvements of recent theorems of Proizvolov and Čoban.

CONTENTS

CHAPTER I

INTRODUCTION

This thesis is a study of various techniques of extending a (continuous) function to an open (continuous) function and applications of these techniques. Several authors, for example Dickman, Krolevec and Whyburn have considered the possibility of extending a continuous mapping to a continuous compact mapping, and to a perfect mapping. Motivated by their work one is led to ask the following question: Is every continuous function from a topological space X into a topological space the restriction of an open continuous function ? This question is answered in the affirmative by giving three different proofs. Many applications of the techniques developed in the process are obtained.

Chapter one gives preliminaries and background results that will be needed throughout the thesis. In chapter two it is shown that a (continuous) function f from a topological space X into a topological space Y can be extended to an open (continuous) function f^* from a topological space X^* onto Y. An attempt has also been made to relate the topological properties of X,Y and the mapping properties of f with the topological properties of X^*. Many results are obtained in this direction.

Chapter three discusses some applications of the results and techniques of chapter two. It has been shown how the results and techniques of chapter two can be utilized to obtain analogues and improvements of various results on open mappings. As an illustration of this fact improvements (or analogues) of recent theorems of Arhangelskiĭ, Čoban and

Proizvolov on finite-to-one mappings are obtained. Similarly improvements (or analogues) of recent theorems of Keesling, Hodel and Nagami on dimension are obtained.

Chapter four deals with two other methods of extending a continuous function to an open continuous function. A modification of the last method is also useful in obtaining partial improvements of recent theorems of Proizvolov and Čoban.

1. Preliminaries and notations

A topological space is a pair $(X,\mathcal{T})$ where X is a nonempty set and $\mathcal{T}$ is a family of subsets of X containing X, the empty set and is closed under arbitrary unions and finite intersections. The members of $\mathcal{T}$ are called open sets. A closed set is the complement of an open set. A set which is both open and closed will be referred to as a clopen set. When confusion is unlikely we will denote $(X,\mathcal{T})$ by simply X. When it is desired to call particular attention to the topology $\mathcal{T}$ of X, or when the underlying point set is to be provided with more than one topology, we shall refer to X as 'the topological space $(X,\mathcal{T})$'. The algebraic ring of continuous real-valued functions will be denoted by $C(X)$, the subring of bounded functions of $C(X)$ will be denoted by $C^*(X)$.

The system of real numbers will be denoted by $\mathbb{R}$, the subsystem of positive integers by $\mathbb{N}$. Throughout the thesis I and J will denote arbitrary indexing sets unless otherwise specified.

As for reference notation, when we refer to a result in the same chapter we state its number only; for example we write - see Proposition 4.6. When we refer to a result in another chapter, we give the number of

the chapter and of the result - Theorem II 2.2 means theorem 2.2 in chapter II.

Definition 1.1. Let X be a topological space. We say that X is a T_0-space in case for each $x,y\in X$ with $x \neq y$, there is an open set containing one of x or y and does not contain the other. We say that X is a T_1-space in case for each $x\in X$ the set $\{x\}$ is closed or equivalently for each $x,y\in X$ with $x \neq y$ there are open sets U and V containing x and y respectively such that $y\notin U$ and $x\notin V$. We say that X is a Hausdorff space or T_2-space in case for each $x,y\in X$ with $x \neq y$ there are disjoint open sets U and V such that $x\in U$ and $y\in V$. We say that X is a functionally Hausdorff space in case for each for each $x,y\in X$ with $x \neq y$ there exists a function $f\in C(X)$ such that $f(x) = 0$ and $f(x) = 1$. We say that X is a regular space in case for each $x\in X$ and each closed set F with $x\notin F$, there are disjoint open sets U and V such that $x\in U$ and $F\subset V$. We say that X is a T_3-space in case X is a regular T_1-space. We say that X is a completely regular space in case for each $x\in X$ and each closed set F with $x\notin F$ there exists a function $f\in C(X)$ such that $f(x) = 0$ and $f(F) = 1$. A completely regular T_1-space will be referred to as a Tychonoff space. We say that X is a normal space in case for each pair F_1,F_2 of disjoint closed sets there are disjoint open sets U and V such that $F_1 \subset U$ and $F_2 \subset V$. A normal T_1-space is called a T_4-space. We say that X is hereditarily normal space in case every subspace of X is a normal space. We say that X is a perfectly normal space in case X is normal and each closed subset of X is a G_δ set.

Let S be a subset of X. We say that S is C-embedded in X if every continuous real-valued function on S extends to a continuous real-valued function on X. We say that S is C*-embedded in X in case every bounded continuous real-valued function on S extends to a bounded continuous real-valued function on X.

Let P be a topological property. We say that X has the property hereditary P in case every subspace of X has property P. For example, we say that X is hereditary normal in case every subspace of X is normal.

Let $f \in C(X)$. Then the set $Z(f) = \{x \in X : f(x)=0\}$ is called the zero-set of f. The complement of $Z(f)$ is called the co-zero set of f.

The closure of a subset A of X will be denoted by $\mathrm{cl}\, A$ and when there is a possibility of confusion by $\mathrm{cl}_X A$. The interior of A will be denoted by $\mathrm{int}\, A$ or $\mathrm{int}_X A$, the boundary of the set A will be denoted by $\mathrm{Fr}\, A$. The complement of a subset B of X will be denoted by $X-B$. The cardinal of X will be denoted by $|X|$.

Definition 1.2. Let $(X,\mathcal{T})$ be a topological space. A base for the topology $\mathcal{T}$ is a subfamily $\mathcal{B}$ of $\mathcal{T}$ such that for each $x \in X$ and each neighbourhood N of x, there is a member $B \in \mathcal{B}$ such that $x \in B \subset N$. The weight of the space X, denoted weight X, is a minimal infinite cardinal number m such that X has a base of cardinality m. A local base at a point $x \in X$ is a family of neighbourhoods such that each neighbourhood of x contains a member of the family. The local weight of the space X, denoted local weight X, is a minimal infinite cardinal number m such that each point $x \in X$ has a local base of cardinality not greater than m. We say that X is a first countable space if the local weight of the space X is $\aleph_0$. A space of weight $\aleph_0$ is said to be a second countable space.

Definition 1.3. Let $\mathcal{U} = \{U_\alpha\}_{\alpha\varepsilon I}$ and $\mathcal{V} = \{V_\beta\}_{\beta\varepsilon J}$ be covers of X. We say that the cover $\mathcal{V}$ is a refinement of $\mathcal{U}$, write $\mathcal{V} < \mathcal{U}$, in case for every $\beta \varepsilon J$, there exists $\alpha \varepsilon I$ such that $V_\beta \subset U_\alpha$. The cover $\mathcal{U}$ is locally finite if for every $x \varepsilon X$ there is a neighbourhood N of x which intersects at most finitely many members of $\mathcal{U}$. A topological space X is said to be m-paracompact if every open cover of X of cardinality at most m has a locally finite open refinement. We say that X is paracompact in case X is m-paracompact for every cardinal number m that is if every open cover of X has a locally finite open refinement. If A is a subset of X, then the star of A with respect to $\mathcal{V}$, written, $St(A,\mathcal{V})$ is the set $\cup\{V_\beta : V_\beta \varepsilon \mathcal{V}$ and $V_\beta \cap A \neq \phi\}$. Let $\mathcal{V}^* = \{St(V_\beta,\mathcal{V})\}_{\beta\varepsilon J}$. If $\mathcal{V}$ is an open cover of X, then $\mathcal{V}^*$ is also an open cover of X. The cover $\mathcal{V}$ is a star refinement of $\mathcal{U}$, written $\mathcal{V}^* < \mathcal{U}$, if $\mathcal{V}^*$ is a refinement of $\mathcal{U}$.

Definition 1.4. Let $\mathcal{U} = \{U_\alpha\}_{\alpha\varepsilon I}$ be a family of subsets of X. We say that the family $\mathcal{U}$ is (1) star-finite if each U_α intersects only finitely many members of $\mathcal{U}$; (2) discrete if every point of X has a neighbourhood which intersects at most ~~finitely many~~ one member of $\mathcal{U}$; (3) pairwise disjoint if $U_\alpha \cap U_\beta = \phi$ for each pair $\alpha,\beta\varepsilon I$ with $\alpha \neq \beta$; (4) point finite if every point of X belongs to at most finitely many members of $\mathcal{U}$. We say that X is strongly paracompact in case X is a Hausdorff space and every open cover of X has an open star finite refinement. We say that X is a collectionwise normal if for every discrete family $\{F_\alpha\}_{\alpha\varepsilon I}$ of closed subsets of X, there is a family $\{G_\alpha\}_{\alpha\varepsilon I}$ of pairwise disjoint open subsets of X such that $F_\alpha \subset G_\alpha$ for all $\alpha\varepsilon I$. We say that X is metacompact if every open cover of X has an open point finite refinement.

Collectionwise normal spaces were first introduced by Bing in [8]. Bing found the concept of collectionwise normality useful in his study of necessary and sufficient conditions for a topological space to be metrizable.

Definition 1.5. Let X be a topological space. We say that X is connected if it cannot be expressed as the union of two non-empty disjoint open subsets. The space X is said to be pathwise (respectively, arcwise) connected if for each pair $x,y \in X$ with $x \neq y$ there exists a continuous function (respectively, homeomorphism) from the unit interval $[0,1]$ into X such that $f(0) = x$ and $f(1) = y$. We say that X is locally (respectively, pathwise, arcwise) connected if each $x \in X$ has a base of (respectively, pathwise, arcwise) connected neighbourhoods.

Definition 1.6. Let X be a topological space. The space X is said to be a sequential space if any subset A of X, which contains the limit of all sequences from A is a closed subset. We say that X is a Fréchet space if for any subset $A \subset X$, $x \in cl_X A$ if and only if there is a sequence in A which converges to x. A chain net is a net whose directed set is a chain. We say that X is a chain net space if any subset A of X, which contains the limit of all chain nets in A, is a closed set. The space X is said to be a c-space if any subset A of X which contains the closure of its countable subsets, is a closed set. We say that X is a k-space if any subset A of X, with $A \cap K$ closed in K for every compact set K in X, is a closed set.

Sequential spaces have been extensively studied by Franklin in [22] and [23], by Arhangelskiĭ and Franklin in [5], and by Dudley in [18]. Franklin showed that sequential spaces are precisely the quotients of metric spaces (see [22]). Chain net spaces were introduced and shown to be precisely the

quotients of ordered spaces by Herrlich in [29]. Moore and Mrówka in [43] introduced the class of c-spaces. For details on k-spaces see ([1] and [59]).

Definition 1.7. Let $(X,\mathcal{T})$ be a topological space and let m be an infinite cardinal number. We say that X is m-compact in case every open cover of X of cardinality at most m has a finite subcover. An $\aleph_0$-compact space will be referred to as a countably compact space. The space X is sequentially compact if every sequence in X has a convergent subsequence. The space X is said to be pseudocompact if $C(X) = C^*(X)$. A Tychonoff space X is said to be realcompact if it can be embedded as a closed subspace in a product of real lines. We say that X is m-Lindelöf if every open cover of X has a subcover of cardinality at most m. An $\aleph_0$-Lindelöf space will be called a Lindelöf space. The space X is said to be m-separable if X has a dense subset of cardinality at most m. An $\aleph_0$-separable space will be referred to as a separable space.

A cardinal number m is said to be measurable if a set X of cardinality at most m admits a $\{0,1\}$ valued measure μ on the set of all subsets of X such that $\mu(\{x\}) = 0$ for all $x \in X$ and $\mu(X) = 1$. By a $\{0,1\}$ valued measure on a set X, we mean a countably additive function defined on the set of all subsets of X, and assuming only the values 0 and 1.

It is not known whether any measurable cardinal exists.

A compactification of a topological space X is a compact space Y in which X is homeomorphically embedded as a dense subspace.

Definitions used but not defined in this thesis may be found in [10], [19] and [39].

This section will end with a short discussion of the Stone-Čech compactification of a Tychonoff space X. This concept is extensively discussed

in chapter 6 of [25] . It is well known that every Tychonoff space admits a Hausdorff compactification βX in which X is C^*-embedded. This space is called the Stone-Čech compactification of X and is unique in the following sense : If Y is a compact Hausdorff space in which X is dense and C^* embedded, then there exists a homeomorphism between Y and βX which is the identity on X.

2. Adjunction spaces

The process of attaching a space X to a space Y by a continuous map f is of great importance in topology. It includes as special cases the cones and the suspension constructions and also the identification of closed sets to the points.

Let $\{X_\alpha\}_{\alpha \varepsilon I}$ be a nonempty family of topological spaces. Without loss of generality we may assume that the family $\{X_\alpha\}_{\alpha \varepsilon I}$ is pairwise disjoint. Let $X = \bigcup_{\alpha \varepsilon I} X_\alpha$. If we endow X with the largest topology relative to which each inclusion mapping i_α from X_α into X is continuous, then the resultant space is called the disjoint topological sum of the family $\{X_\alpha\}_{\alpha \varepsilon I}$ and denoted by $\bigoplus_{\alpha \varepsilon I} X_\alpha$.

Let f be a continuous function defined on a closed subspace A of X into Y. Let $W = X \oplus Y$. Introduce a relation $\sim$ in W as follows. Let x and y be any two points in the space W. Then $x \sim y$ if and only if any one of the following conditions is satisfied.

(i) $x = y$, (ii) $x = f(y)$, (iii) $y = f(x)$, (iv) $f(x) = f(y)$.

Then $\sim$ is an equivalence relation in the space W. The quotient space Z of the space W over this equivalence relation $\sim$ is called the adjunction space obtained by adjoining X to Y by means of the given map f, denoted by $X \cup_f Y$. Let p be the quotient map of W onto Z.

Then the following is true.

Theorem 2.1. (i) *The mapping* p/Y *is a homeomorphism and the set* $p(Y)$ *is a closed subset of the space* Z. (ii) *The mapping* $p/X-A$ *is a homeomorphism and the set* $p(X-A)$ *is an open subset of the space* Z.

For a proof of the theorem 2.1 see ([19], page 128 or [33], page 122).

Let P be a topological property. We say that the adjunction space Z *preserves the property* P if Z has property P whenever both X and Y have the property P.

Theorem 2.2. *The adjunction space* Z *preserves each of the following properties.*

(1) *Lindelöf property* (2) *normality* (3) *hereditary normality* (4) *paracompactness* (5) *m-paracompactness and normality, where* m *is an finite cardinal number* (6) *perfect normality.*

The above theorem for normality and paracompactness was first proved by Hanner in [26] and [27] respectively. The proof in case of hereditary normality is due to Iséki (see [35]). For an available proof of the theorem see ([34], page 15 and [55]).

The following theorem is due to McCandless (see [42] Theorem 1).

Theorem 2.3. *The adjunctinn space* Z *preserves the property of being* :

(1) *hereditarily paracompact* (2) *hereditarily collectionwise normal* (3) *hereditarily m-paracompact and normal, where* m *is an infinite cardinal number* (4) *hereditarily Lindèlöf.*

3. *The Embedding Lemma.*

One of the problems that has had considerable interest in the study of topological spaces has been that of homeomorphically embedding a given

space X as a dense subspace of a topological extension space Y where Y possesses some desired topological property P, such as compactness, completeness or realcompactness.

A property P is said to be <u>closed-hereditary</u> in case every closed subspace of a space with property P also possesses property P. It is said to be <u>productive</u> in case an arbitrary product of spaces with property P also possesses property P.

A well known procedure employed in attacking the above problem for a property P that is closed hereditary and productive is to embed the given space X homeomorphically into a product of spaces, each possessing the property P, and then to take the closure of that homeomorphic image in the product space as the desired extension space Y. In [39] Kelley establishes an embedding lemma which turns out to be useful for employing the above technique. Indeed it is used to obtain compactifications of Tychonoff spaces and a metrization theorem for second countable T_3-spaces. (see [39], Lemma 4.5, Theorem 4.16, and Theorem 5.24 for the details). In [48] Mrowka proves a slight generalization of the embedding lemma of Kelley which is foundational with respect to the problem under discussion. Moreover, the embedding lemma provides a natural setting for introducing the concept of realcompact spaces. We begin with the statement of the embedding lemma although the proof is omitted (see [39], page 116).

Let X be a topological space and let $\{X_\alpha\}_{\alpha\varepsilon I}$ be a nonempty family of topological spaces. For each $\alpha\varepsilon I$, let f_α be an arbitrary continuous function from X into X_α. Let F denote the family $\{f_\alpha : \alpha\varepsilon I\}$. Then there is associated with the family F, a natural mapping e from X into the product space $\Pi(X_\alpha : \alpha\varepsilon G)$ defined by $e(p) = (f_\alpha(p))_{\alpha\varepsilon I}$. The mapping

e is called the evaluation mapping associated with F. The family F distinguishes points if for each pair of distinct points x and y there is f_α in F such that $f_\alpha(x) \neq f_\alpha(y)$. The family F is said to distinguish points and closed sets if for each closed subset A of X and each point $x \in X-A$ there is an f_α in F such that $f_\alpha(x)$ does not belong to the closure of $f_\alpha(A)$.

Theorem 3.1. (The Embedding Lemma). *If X, $X_\alpha (\alpha \in I)$, F and e are given as in the preceding paragraph, then the following statements are true.*

(1) *The evaluation mapping e is continuous if and only if each f_α is continuous.*

(2) *The function e is an open function from X onto $e(X)$ if F distinguishes points and closed sets.*

(3) *The mapping e is one-to-one if and only if the family F distinguishes points.*

The importance of the embedding lemma is that it reduces the problem of embedding a topological space X homeomorphically into a product space $\Pi(X_\alpha : \alpha \in I)$ to that of finding a 'rich enough' family of continuous functions from X into each X_α. Three well known applications of the embedding lemma are Uryshon's metrization theorem, the Stone-Čech compactification, and the completion of a Hausdorff Uniform space.

4. Finite-to-one open mappings

Finite to one mappings form an important class of mappings and have been extensively studied by several authors. In this section we restrict ourselves to the subclass of finite-to-one open mappings. In fact we confine ourselves to the results which say something about the inverse

image of open (clopen) finite-to-one mappings. As far as we know the results of this sort were first investigated by Proizvolov (see [53]) and later on by Arhangelskiǐ (see [3], [4]), and Čoban (see [14]). In the discussion that follows all of the spaces are *assumed to be Hausdorff and all maps are assumed to be continuous onto* unless explicitly stated otherwise. The following result is due to Proizvolov (see [53] , Theorems 1, 1' and their corollaries) and will be referred to later on.

Theorem 4.1. *Let* f *be an open continuous finite-to-one mapping of a Hausdorff space* X *onto a Hausdorff space* Y. *The following statements are valid*.

(a) *If* X *is locally compact, then weight* $X \leq$ *weight* Y.

(b) *If* X *is m-compact, then weight* $X \leq$ *weight* Y.

(c) *If* X *is a metric space, then weight* $X \leq$ *weight* Y.

(d) *If* X *is complete in the sense of Čech, then weight* $X \leq$ *weight* Y.

Note that a Tychonoff space X is *complete in the sense of Čech* if X is a G_δ set in its Stone-Čech compactification βX.

A mapping which is both open and closed will be referred to as a *clopen mapping*.

Arhangelskiǐ showed that the hypothesis on X in Proizvolov's results is unessential in the case of closed mappings (see [4], Theorem 2). Specifically, he proved the following result.

Theorem 4.2. *Let* f *be an open continuous finite-to-one mapping of a Hausdorff space* X *onto a Hausdorff space* Y. *If* f *is also a closed mapping, then the weight of* X *is equal to the weight of* Y.

Arhangelskiǐ continued his studies on open finite-to-one mappings in [3] and obtained the following as a main result.

Theorem 4.3. Let f be a clopen continuous finite-to-one mapping of a Tychonoff space X onto a topological space Y. If Y is metrizable, then X is also metrizable.

A Tychonoff space X is called a p-space (plumbed space) if there exists a sequence $\{\gamma_n\}_{n\in N}$ of collections of open sets in βX such that each γ_n covers X and $\bigcap_{n=1}^{\infty} St(x,\gamma_n) \subseteq X$ for any $x \in X$.

The next theorem is due to Čoban and is theorem 2 of [14], and will be referred to later on.

Theorem 4.4. Let f be an open continuous finite-to-one mapping of a p-space X onto a topological space Y. If Y is metrizable, then X is metrizable.

The following result is due to Čoban (see [14], Theorems 2 and 4).

Theorem 4.5. Let f be an open continuous finite-to-one mapping of a Hausdorff space X onto a Hausdorff space Y. Then X is hereditarily metacompact (respectively, Lindelöf) if and only if Y is hereditarily metacompact (respectively, Lindelöf).

The next theorem is due to Proizvolov and is theorem 3 of [14].

Theorem 4.6. Let f be an open continuous finite-to-one mapping of a T_3-space X onto a Hausdorff space Y. If Y is hereditarily paracompact, then X is also a hereditarily paracompact space.

5. Dimension of topological spaces

Dimension is an important unifying concept in topology and is a generalization of the notion of the dimension of Euclidean spaces. To certain topological spaces an integer not less than -1 is assigned. The number -1 is only assigned to the empty space. To those spaces whose

∞. The dimension function can be defined in many ways. We shall consider three kinds of dimension of a topological space : ind X, Ind X and dim X which are defined below. For a detailed study of these concepts, an interested reader is referred to [51]. Here we assume that <u>for the purposes of dimension the empty space ϕ is also a topological space.</u>

<u>Definition</u> 5.1. Let X be a T_3-space and let n denote a non-negative integer. We say that

(MU1) ind $X = -1$ if and only if $X = \phi$

(MU2) ind $X \leq n$ if for every point $x \in X$ and every neighbourhood $V \subseteq X$ of x there exists an open set $U \subseteq X$ such that $x \in U \subseteq cl_X U \subseteq V$ and ind Fr $U \leq n-1$.

(MU3) ind $X = n$ if ind $X \leq n$ is true and ind $X \leq n-1$ is false.

(MU4) ind $X = \infty$ if ind $X \leq n$ is false for every n.

Conditions (MU1) - (MU4) assign to every T_3-space X an integer ind X greater than -2 or the "infinity number" ∞. The number ind X is called the <u>small inductive dimension</u> of the space X.

It follows immediately from the definition of the small inductive dimension of a T_3-space X that ind $X \leq n$ if and only if there exists a base $\mathcal{B}$ in X such that ind Fr $U \leq n-1$ for every $U \in \mathcal{B}$. In particular, ind $X = 0$ if and only if X has a base of clopen sets.

<u>Definition</u> 5.2. Let X be an arbitrary T_4-space and let n denote a non-negative integer. We shall say that

(BC1) Ind $X = -1$ if and only if $X = \phi$.

(BC2) Ind $X \leq n$ if for every closed set $A \subseteq X$ and every open set $V \subseteq X$ containing A there is an open set $U \subseteq X$ such that $A \subseteq U \subseteq cl_X U \subseteq V$ and Ind Fr $U \leq n-1$.

(BC3) Ind $X = n$ if Ind $X \leq n$ is true and Ind $X \leq n-1$ is false.

(BC4) Ind $X = \infty$ if Ind $X \leq n$ is false for every n.

Conditions (BC1) - (BC4) assign to every T_4-space X an integer Ind X greater than -2 or the " infinite number " ∞. The number Ind X is called the large inductive dimension of the space X.

Definition 5.3. Let X be a topological space and let $\mathcal{U}$ denote a collection of subsets of X. We say that the collection $\mathcal{U}$ is of order n, and write Ord $\mathcal{U} = n$, if n is the greatest integer satisfying the condition : the family $\mathcal{U}$ contains $n+1$ sets which have a nonempty intersection. If Ord $\mathcal{U} \leq n$, then for every $n+2$ distinct elements $A_1, \ldots, A_{n+2}$ of the family $\mathcal{U}$ we have $A_1 \cap A_2 \cap \ldots \cap A_{n+2} = \phi$.

In particular, a family of order -1 can contain only the empty set, and a family of order 0 consists of disjoint sets.

Definition 5.4. Let X be a Tychonoff space and let n denote an integer greater than -2. We shall say that

(CL1) dim $X \leq n$ if every co-zero set cover of the space X has a finite co-zero set cover refinement of order $\leq n$.

(CL2) dim $X = n$ if dim $X \leq n$ is true and dim $X \leq n-1$ is false.

(CL3) dim $X = \infty$ if dim $X \leq n$ is false for every n.

Conditions (CL1) - (CL3) assign to every Tychonoff space X an integer dim X greater than -2 or the " infinite number " ∞. The number dim X is called the covering dimension of the space X.

It follows from the definition of the covering dimension that dim $X = -1$ if and only if $X = \phi$.

Below we state a result without proof, which will be useful in the sequel.

Theorem 5.5. *If* M *is a subspace of a Tychonoff space* X *such that* M *is* C^*-*embedded in* X, *then* $\dim M \leq \dim X$.

In particular, if M is a closed subspace of a T_4-space X, then $\dim M \leq \dim X$.

For a proof of theorem 5.5 (see [20], page 268).

6. Open functions and dimension.

The purpose of this section is to give a brief survey of the results on open functions and dimension with particular emphasis on the results that will be of use in the sequel. Suppose f is an open function from a metric space X onto a metric space Y. We are interested in the relation between $\dim X$ and $\dim Y$. Let us quickly review the known dimension theoretical results in this direction. Alexandroff (see [41], page 48) proved that if X is a compact metric space, Y a metric space, and f an open continuous at most countable-to-one function from X onto Y, then $\dim X = \dim Y$. Roberts, in 1947 (see [54]) improved upon this result in one direction by showing that if X is a separable metric space, Y a locally compact separable metric space, and f an open continuous function from X onto Y such that for each point $y \in Y$, $f^{-1}(y)$ is not dense in itself, then $\dim Y \leq \dim X$. In 1963, Hodel (see [31]) obtained a number of modifications of these results of Alexandorff and Roberts. In the same paper he also obtained a generalization of Alexandroff's result. Below we give the results due to him which will be of use later on. The following result is a theorem 2.5 of [31].

Theorem 6.1. *Let* X *and* Y *be metric spaces and let* f *be an open function from* X *onto* Y *such that for each point* $y \in Y$, $f^{-1}(y)$ *is discrete. If* Y *is locally compact and separable, then* $\dim X \leq \dim Y$.

Hodel himself gave the examples to show that the hypotheses of separability and local compactness cannot be dropped in the above theorem. Moreover, he proved that hypotheses on Y are unessential if the function f is continuous (see [31], Theorem 2.9).

Keesling, in 1968 (see [36], Theorem II.1) generalized the above mentioned result of Hodel and obtained the following result.

Theorem 6.2. *Let* f *be an open continuous function from a metric space* X *onto a metric space* Y *such that* $f^{-1}(y)$ *is discrete for every* $y \in Y$. *Then for all subset* $K \subseteq X$, *dim* $K \leq$ *dim* $f(K)$.

The following result is due to Hodel (see [31], Theorem 4.1).

Theorem 6.3. *If* X *is a metric space*, Y *a locally compact separable metric space, and* f *a clopen function from* X *onto* Y *such that for each* $y \in Y$, $f^{-1}(y)$ *is discrete, then dim* $X =$ *dim* Y.

The next theorem is also due to Hodel (see [31], Theorem 5.1) and is a generalization of Alexandroff's theorem (that was mentioned in the beginning of this section).

Theorem 6.4. *Let* X *and* Y *be metric spaces and let* f *be an open continuous countable-to-one function from* X *onto* Y. *If* X *is locally compact, then dim* $X =$ *dim* Y.

Keesling in [36] pointed out a class of functions which preserve dimension on closed subsets. Before discussing his results we need the following definition.

Let f be a function from a topological space X into a topological space. We say that X is *σ-closed* in case there exists a countable closed cover $\{A_i\}_{i\in \mathbb{N}}$ of X such that for each i the set $f(A_i)$ is closed in Y and the restriction f/A_i is a closed function.

The σ-closed mappings arise naturally as (1) continuous mappings of σ-compact spaces, (2) finite-to-one mappings, and (3) open mappings such that the inverse images of points are discrete on separable metric spaces.

Keesling in [36] showed that the following types of continuous functions preserve dimension on closed subsets : (1) f open with $f^{-1}(y)$ discrete and X separable ; (2) f finite-to-one and open ; (3) f open, σ-closed, $f^{-1}(y)$ discrete for all $y \in Y$; and (4) f exactly k-to-one and σ-closed. The following result is due to him (see [36], Theorems II.3, II.4 and II.7).

Theorem 6.5. *Let X and Y be metric spaces and let f be an open, continuous mapping of X onto Y. Then the following statements hold.*

(a) *If the mapping f is σ-closed and if $f^{-1}(y)$ is discrete for all $y \in Y$, then f is dimension preserving on closed subsets.*

(b) *If the space X is also separable and if $f^{-1}(y)$ is discrete for all $y \in Y$, then f is dimension preserving on closed subsets.*

(c) *If f is also finite-to-one, then f is dimension preserving on closed subsets.*

Theorem 6.6. *Let X and Y be Tychonoff spaces and let f be an open continuous mapping of X onto Y. Then the following statements are true.*

(a) *If f is also a closed mapping, then* $\dim X = \dim Y$.

(b) *If the space X is also normal and if f is a closed mapping, then f is dimension preserving on closed subsets.*

(c) *If the spaces X and Y are paracompact and if the mapping f is finite-to-one, then* $\dim X = \dim Y$.

(d) *If the spaces X and Y are hereditarily paracompact and if the mapping f is finite-to-one, then* $\operatorname{Ind} X = \operatorname{Ind} Y$.

The statements (a) and (b) of the above theorem are due to Keesling (see [38], Theorems III.1 and III.2) and statements (c) and (d) are due to Nagami (see [49], Theorem 4.1).

7. Closed coverings and spaces having weak topology with respect to a closed covering

Let X be a topological space and let $\{A_\alpha\}_{\alpha\in I}$ be a closed covering of X. We shall say that X has the *weak topology with respect to* $\{A_\alpha\}_{\alpha\in I}$ if for any subset $J \subseteq I$, the union A_J of the family $\{A_\beta\}_{\beta\in J}$ is closed in X and if any $S \subseteq A_J$ for which $S \cap A_\beta$ is an open set in A_β (relative to the subspace topology of A_β) for each $\beta \in J$ is necessarily open in the subspace A_J; the word 'open' may of course be replaced by 'closed'.

According to this definition any CW-complex K in the sense of J.H.C. Whitehead has the weak topology with respect to the closed covering which consists of the closures of all the cells of K and any topological space X has the weak topology with respect to each of its locally finite closed coverings.

The above definition of weak topology is due to Morita [45] and slightly differs from the usual definition (see [19], page 131).

The following result is due to Morita (see [45], Theorems 2 and 3) and will be referred to later on.

Theorem 7.1. *Let X be a topological space having the weak topology with respect to a closed covering $\{A_\alpha\}_{\alpha\in I}$. If each A_α is normal (respectively, collectionwise normal, hereditarily normal, perfectly normal), then X is normal (respectively, collectionwise normal, hereditarily normal, perfectly normal). Furthermore, if $\dim A_\alpha \leq n$ for each α, then $\dim X \leq n$.*

The next theorem is also due to Morita (see [46], Theorems 1 and 2).

Theorem 7.2. *Let X be a topological space having the weak topology with respect to a closed covering $\{A_\alpha\}_{\alpha\varepsilon I}$ and let m be an infinite cardinal number. If each A_α is m-paracompact and normal, then X is m-paracompact and normal.*

The following theorem is due to Hodel (see [32], Theorem 5.1).

Theorem 7.3. *Let X be a topological space and let $\{F_\alpha\}_{\alpha\varepsilon I}$ be a locally finite closed covering of X. If each subspace F_α is metacompact, then X is also metacompact.*

A topological space X is said to be a *stratifiable space* if X is T_1 and to each open set $U \subseteq X$, one can assign a sequence $\{U_n\}_{n\varepsilon N}$ of open subsets of X such that (a) $cl_X U_n \subseteq U$, (b) $\bigcup_{n=1}^{\infty} U_n = U$, (c) $U_n \subseteq V_n$ whenever $U \subseteq V$.

Ceder in [13] first introduced the class of stratifiable spaces. He initiated the study of T_3-spaces which have σ-closure preserving base and called then *M_1-spaces*. Ceder also introduced *M_i-spaces for* $i = 2, 3$. His M_3-spaces are precisely the stratifiable spaces. The later nomenclature is due to Borges [9]. Every M_1-space is a stratifiable space and an M_1-space need not be metrizable. For example, let $\mathbb{N}$ be the space of integers with discrete topology and let $\beta\mathbb{N}$ be its Stone-Cech compactification. Let $X = \mathbb{N} \cup \{p\}$ where $p \varepsilon \beta\mathbb{N} - \mathbb{N}$. Then X has a σ-closure preserving base but X is not first countable at p. (see for example, Corollary 9.6 of [25]). Thus X is not metrizable. Ceder among many other results on stratifiable spaces obtained the following result (see [13], Theorem 7.2).

Theorem 7.4. *Let X be a T_1-space and let $\{F_\alpha\}_{\alpha\varepsilon I}$ be a closed covering of X. If each subspace F_α is a stratifiable space, then X is also a stratifiable space.*

The next theorem is due to Nagata [50].

Theorem 7.5. Let X be a topological space and let $\{F_\alpha\}_{\alpha\varepsilon I}$ be a locally finite closed covering of X. If each subspace F_α is metrizable, then X is also metrizable.

8. Brief survey of the results on closed and perfect extensions of maps.

G.T. Whyburn in 1953 (see [56]) developed a unification procedure for the domain and range of a given mapping which achieved a meaningful compact extension of the mapping. In this way any mapping between Hausdorff spaces was exhibited as a partial mapping (restriction) of a compact mapping. We briefly mention his method here.

8.1. Let X and Y be Hausdorff spaces and let f be a continuous map of X into Y. Without any loss of generality one may assume that X and Y are disjoint (for otherwise take disjoint copies of X and Y). Whyburn defined a unified space for the mapping in the following way.

Let W denote the set theoretic union of X and Y. Define a set Q in W to be open if it satisfies the following conditions :

(i) The sets $Q \cap X$ and $Q \cap Y$ are open in X and Y respectively.

(ii) For any compact set K in $Q \cap Y$, the set $f^{-1}(K) \cap (X-(Q \cap X))$ is a compact set in X.

The set W together with the collection of open sets so defined is a T_1-topological space which is called the unified space of f and which we denote by Z. The injections of X and Y into Z are open and closed respectively. Thus X is embedded in Z as an open set and Y is embedded in Z as a closed set.

Associated with Z is a retraction r from Z onto Y defined by $r(z) = f(z)$ for $z \varepsilon X$ and $r(z) = z$ for $z \varepsilon Y$. This retraction is

continuous and compact. Thus when Y is (locally) compact, Z is also (locally) compact. Furthermore, if f is an open mapping, so also is r.

In [56] it is shown that when X and Y are both locally compact Hausdorff spaces, so also is Z and that when X and Y are both locally compact separable metric spaces, Z is also a locally compact separable metric space.

Let $\tilde{E}$ denote the closure in Z of any subset E of Z. The restriction r/X of r to X is topologically equivalent to f and thus $r/\tilde{X}$ is a compact mapping which extends f.

Heinz Baur in his study of conservative maps (see [6]) showed that for every continuous mapping f from a locally compact Hausdorff space X onto a locally compact Hausdorff Y, there exists a locally compact Hausdorff space X_o and a continuous mapping f_o satisfying the following :

(i) X is a dense subset of X_o.

(ii) f_o/X is topologically equivalent to f.

(iii) f_o is a compact mapping.

(iv) f_o is one-to-one on $X_o - X$.

Furthermore, he showed that every space satisfying (i) - (iv) is homeomorphic to X_o under a homeomorphism that leaves points of X fixed. Thus his X_o is homeomorphic to Whyburn's $\tilde{X}$ and f_o is topologically equivalent to $r/\tilde{X}$.

In [15] Dickman extensively studied the unified space of a mapping and related topological properties of X, Y and mapping properties of f with the topological properties of Z. He showed that Z is paracompact if and only if Y is paracompact. He also gave a necessary and sufficient condition for the metrizability of Z and obtained a bound for the covering

dimension of Z in terms of the strong inductive dimension of Y and the covering dimension of point inverses of f. He also showed that the domain of any compact retraction is a unified space of a mapping. He also investigated the local connectedness of Z in terms of mapping properties of f. Furthermore, he showed that several interesting maps have a locally connected unified space.

8.2. A continuous mapping f from a topological space X into a topological space Y is said to be <u>locally perfect</u>, if for each $x \in X$ there is a neighbourhood U of x such that $f(\mathrm{cl}\, U)$ is closed in Y and $f/\mathrm{cl}\, U$ is a perfect map. Any continuous map from a locally compact Hausdorff space into a Hausdorff space is locally perfect. In particular any mapping of a discrete space into a Hausdorff space is locally perfect. N. Krolevec, in 1967 (see [40]), showed that for every locally perfect map f from a T_1-space X onto a T_1-space Y, there is a superspace $\tilde{X}$ of X and a perfect mapping $\tilde{f}$ from $\tilde{X}$ onto Y whose restriction to X is f. Here we briefly describe his method of construction.

Let f be a locally perfect map from a T_1-space X onto a T_1-space Y. Let Y^* denote the set of all points $y \in Y$ such that either $f^{-1}(y)$ is not compact or f is not closed at y(We say that f is <u>not closed at a point</u> $y \in Y$ if there exists a closed set F in X such that $y \in f(F)$ and $f(F)$ is not closed in Y). Let $\tilde{X}$ denote the disjoint set theoretic union of the sets X and Y^*. Let $\tilde{f}$ be the function from $\tilde{X}$ onto Y defined by $\tilde{f}(z) = f(z)$ if $z \in X$ and $\tilde{f}(z) = z$ if $z \in Y^*$. Define a topology on $\tilde{X}$ as follows : a basic neighbourhood of a point $x \in X$ remains the same, If $x \in \tilde{X}-X$, its neighbourhood will be the set of the form $U = \tilde{f}^{-1}(V)-F$, where V is a neighbourhood of the point $y = \tilde{f}(x)$ in Y, $F \subset X$ and closed in X,

$f(F)$ is closed in Y and the mapping f/F is perfect. Then $\tilde{f}$ is a perfect mapping. The space $\tilde{X}$ and the map $\tilde{f}$ satisfy the following.

(i) $\tilde{f}/X = f$

(ii) $\tilde{f}(\tilde{X}-X)=Y^*$

(iii) X is dense in $\tilde{X}$

(iv) $\tilde{f}$ is one-to-one on $\tilde{X}-X$.

Krolevec showed that these hypotheses determine $\tilde{X}$ and $\tilde{f}$ uniquely. Furthermore, he related the topological properties of X and Y with the topological properties of $\tilde{X}$. He showed that if X and Y are Hausdorff spaces or Tychonoff spaces or locally compact Hausdorff spaces, then so is $\tilde{X}$. He also proved that if X and Y are separable metric spaces, then $\tilde{X}$ is metrizable. Moreover, if $\tilde{f}_1$ from $\tilde{X}_1$ onto Y is any perfect extension of f, then there is a continuous mapping $\tilde{h}$ from $\tilde{X}_1$ onto $\tilde{X}$ such that $\tilde{h}(x) = x$ for each $x \varepsilon X$ and $\tilde{f}_1=\tilde{f} \circ h$.

8.3. In 1969, Dickman (see [16]) extended the results of Whyburn and Krolevec on compact and perfect extensions of maps. He modified Whyburn's technique of unifying the domain and range of a mapping to show that any continuous mapping from a topological space into another is the restriction of a perfect mapping. Before mentioning his method of construction, we shall need the following definition.

A filter base M is said to be <u>directed toward a set</u> A if any filter base finer than M has a cluster point in A.

Let X and Y be topological spaces and let f be a function from X into Y. Without any loss of generality one may assume that X and Y are disjoint (for otherwise just take disjoint copies of X and Y). Let W denote the union of X and Y. Define a set Q in

W to be open if and only if it satisfies the following conditions.

(i) The sets $Q \cap X$ and $Q \cap Y$ are open in X and Y respectively.

(ii) For any filter base $\mathcal{M}$ in $Q \cap Y$ converging to a point $y \in Q \cap Y$ (relative to Y) $f^{-1}(M) \cap (X-Q \cap Y)$ is empty for some $M \in \mathcal{M}$, or else the filter base $\{f^{-1}(M) \cap (X-Q \cap X): M \in \mathcal{M}\}$ is directed toward $f^{-1}(y)$.

The set W together with the collection of open sets so defined is a topological space. Associated with W is a retraction r from W onto Y defined by $r(z) = z$ for $z \in Y$ and $r(z) = f(z)$ for $z \in X$. This retraction is closed and compact. Thus when Y is (locally) compact, Z is also (locally) compact. Furthermore, if f is continuous or open, so also is r.

The closure of r in W will be called the <u>r-extension of</u> X and denoted by X_o. We call r/X_o the <u>r-extension of</u> f and denote it by f_o. The extensions X_o and f_o of X and f, respectively satisfy the following properties :

(i) The space X is an open dense subset of X_o,

(ii) The function f_o is a closed and compact function,

(iii) The function f_o restricted to X^* is a one-to-one, where X^* is the closure of X_o-X in X_o

(iv) a subset K of X_o is closed in X_o if and only if $K \cap X$ and $K \cap X^*$ are closed in X and X^*, respectively, and f_o/K is a closed and compact function.

(v) a closed subset K of X is closed in X_o if and only if f/K is a closed and compact function.

Furthermore, the r-extensions X_o and f_o of X and f, are unique extensions satisfying properties (i) - (iv). If X and Y are T_1-spaces so also is X_o. If f is continuous and X and Y are locally compact Hausdorff spaces (respectively, locally compact separable metric spaces), then X_o is a locally compact Hausdorff space (respectively, locally compact separable metric space). Dickman also showed that if f is any continuous map of a topological space X into a regular space Y, then there exists topologically unique extensions X_o and f_o of X and Y respectively having properties (i) - (iii) and (v). In the same paper he also pointed out the following uniqueness theorem for Whyburn's unified space extensions : Let f be a continuous mapping of a topological space X into a regular space Y. Then there exists topologically unique extensions $\tilde{X}$ and $\tilde{f}$ of X and f respectively, having properties (i), (iii) as well as (ii)' $\tilde{f}$ is a compact mapping, and (v)' a closed subset K of X is closed in $\tilde{X}$ if and only if f/K is a compact mapping.

In general, the two concepts, unified space extensions and r-extensions donot coincide. In case f is a continuous map of X into Y and the spaces X, Y are locally compact Hausdorff spaces, then Baur's extension, unified space extension and r-extension all the three extensions coincide.

8.4 A <u>compactification of a continuous mapping</u> f from a Hausdorff space X onto a Hausdorff space Y is a pair (X^*, f^*) where X is a Hausdorff space containing X as a dense subspace, f^* is a perfect mapping of X^* onto Y such that $f^*/X = f$. Thus G.T. Whyburn introduced the notion of a mapping compactification in 1953 when he showed that every continuous mapping of one locally compact Hausdorff space onto another is

a partial mapping of a perfect mapping on a Hausdorff space (see [57],[58]). He also pointed out in [58] that a compactification for a continuous function from a Tychonoff space X onto a Tychonoff space Y can be obtained by restricting the extension f^β to $f^{\beta^{-1}}(Y)$, where f^β is the extension of f to βX.

In 1969, G.L. Cain Jr (see [12]) studied some general properties of mapping compactifications and constructed a so called filter space compactification of a Tychonoff space onto a T_3-space. Each of these compactification is associated in a natural way with a compactification of the domain of f. For Y a locally compact Hausdorff he showed that the domain X is a Tychonoff space if and only if f has a compactification and if X is a Tychonoff space, every compactification of f is a filter space compactification.

CHAPTER II

A METHOD OF CONSTRUCTING OPEN EXTENSIONS AND ITS PROPERTIES

1. In this chapter we show that any function (respectively, any continuous function) f from a topological space X into a topological space Y is the restriction of an open function (respectively, an open continuous function) f^* from a topological space X^* onto Y. The space X^* is obtained as a quotient of a disjoint topological sum of X and copies of Y. If X and Y are T_1-spaces, then the space X^* can also be obtained as an adjunction space. We also relate the topological properties of X, Y and the mapping properties of f with the topological properties of X^*. It turns out that X^* is a T_i-space (where $i = 0,1,2,3,3\frac{1}{2},4$) if and only if X and Y are T_i-spaces. Also the paracompactness of X and Y is carried over to X^* and as well as the various forms of normality. We also give a necessary and sufficient condition for the space X^* to be metrizable (respectively, first countable, second countable, locally compact) in terms of the topological properties of X, Y and the mapping properties of f. The compactness properties of X, Y and the mapping properties of f are also related with the compactness properties of X^*. An attempt has also been made to relate the mapping properties of the function f with the mapping properties of the function f^*.

2. Open extensions

Let f be a function, not necessarily continuous, from a topological space X into a topological space Y. In this section we

construct a superspace X^* of X and an open function f^* from X^* onto Y whose restriction to X is f such that the function f^* is continuous, if f is.

Definition 2.1. We shall call a point $x \in X$ and its image $f(x) \in Y$ singular point of X and Y respectively, if there is an open set U containing x such that $f(U)$ is not a neighbourhood of $f(x)$.

Let $S_f(X)$ and $T_f(Y)$ denote the set of singular points of X and Y respectively. Throughout this thesis $S_f(X)$ and $T_f(Y)$ will carry the same meaning, unless expressly stated otherwise. When confusion is unlikely, we shall denote $S_f(X)$ by S and $T_f(Y)$ by T. When it is desired to call particular attention to the function f, or when more than one function is involved in the discussion, we shall refer to S by $S_f(X)$ and T by $T_f(Y)$.

It is immediate from the definition that : the function f is open if and only if the set $S_f(X)$ of singular points of X is empty (or equivalently the set $T_f(Y)$ of singular points of Y is empty).

The following theorem is fundamental in this chapter and many applications of this will be obtained in the next chapter.

Theorem 2.2. Let f be a function from a topological space X into a topological space Y. Then there is a superspace X^* of X and an open function f^* from X^* onto Y whose restriction to X is f. Moreover, the function f^* is continuous, if f is continuous.

Proof. For each singular point x of X, let Y_x be a copy of Y. Let $W = X \oplus (\oplus Y_x)$, where the second disjoint topological sum is taken over all singular points of X. Define a relation $\sim$ in W as follows :

$x_1 \sim x_2$ if and only if any one of the following conditions is satisfied.

(1) $x_1 = x_2$

(2) $x_1 = f(x_2)$ and $x_1 \varepsilon Y_{x_2}$

(3) $x_2 = f(x_1)$ and $x_2 \varepsilon Y_{x_1}$

Then it is straightforward to verify that $\sim$ is an equivalence relation in the space W. Let X^* be the quotient space of W over this equivalence relation $\sim$ and let q be the quotient map (natural projection) from W onto X^*. The inclusion map i_X from X into W composes with the quotient map q to give a homeomorphic embedding of X into X^*. Hence X^* may be thought of as an extension of X.

Let f_1 be a function from W onto Y whose restriction to X is f and whose restriction to each Y_x is the identity map 1_Y from Y_x onto Y. Define a function f^* from X^* onto Y by $f^*(p) = f_1(q^{-1}(p))$ for each $p \varepsilon X^*$. Then f^* is a unique function satisfying $f^* \circ q = f_1$. For let f_2 be any function from X^* into Y satisfying $f_2 \circ q = f_1 = f^* \circ q$. Since q is an onto mapping, for each $p \varepsilon X^*$ there is a $z \varepsilon W$ such that $q(z) = p$. Therefore, $f_2(p) = f_2(q(z)) = f_2 \circ q(z) = f^* \circ q(z) = f^*(q(z)) = f^*(p)$. The restriction of f^* to $q(X)$ is essentially f.

To show that f^* is an open function let V be an open set X^*. Then $q^{-1}(V)$ is an open set in W and

$$q^{-1}(V) = (q^{-1}(V) \cap X) \cup (q^{-1}(V) \cap (\bigoplus_{x \varepsilon S} Y_x))$$

$$= (q^{-1}(V) \cap X) \cup (\bigcup_{x \varepsilon S} (q^{-1}(V) \cap Y_x))$$

Now

$$f^*(V) = f_1(q^{-1}(V))$$

$$= f_1((q^{-1}(V) \cap X) \cup (\bigcup_{x \in S} (q^{-1}(V) \cap Y_x)))$$

$$= f_1(q^{-1}(V) \cap X) \cup (\bigcup_{x \in S} f_1(q^{-1}(V) \cap Y_x))$$

$$= f\ (q^{-1}(V) \cap X) \cup (\bigcup_{x \in S} f_1(q^{-1}(V) \cap Y_x)) \quad \ldots \quad (1)$$

Since $q^{-1}(V)$ is an open set in Y_x and since the function f_1 restricted to Y_x is the identity map, the second term on the right hand side of (1) is the union of open sets and hence it is open. Further if the set $f(q^{-1}(V) \cap X)$ is also open in Y, then the set $f^*(V)$ being the union of open sets is open. If the set $f(q^{-1}(V) \cap X)$ is not open, say it is not a neighbourhood of one of its points y, then $y = f(x)$ for some $x \in q^{-1}(V) \cap X$ and x is a singular point of X. Since $q^{-1}(V)$ is a saturated open set, $q^{-1}(V)$ contains a neighbourhood of $f(x)$. Hence $f^*(V)$ being a neighbourhood of each of its points is an open set.

Since a function from a quotient space into a topological space is continuous if and only if its composition with the quotient map is continuous, it follows that the function f^* is continuous if and only if the composition map $f^* \circ q = f_1$ is continuous. Hence the function f^* is continuous if and only if the function f is continuous. This proves the last assertion of the theorem. Thus the proof of the theorem is complete.

Dickman in [16] (see I.8.3) showed that any function (respectively, any continuous function) f from a topological space X into a topological

space Y is the restriction of a closed compact function (respectively, a perfect mapping) $\overline{f}$ from a topological space $\overline{X}$ onto Y. Moreover, the function $\overline{f}$ is open, if the function f is open. Hence together with the theorem 2.2 we obtain the following result.

Corollary 2.3. Let f be a function from a topological space X into a topological space Y. Then there is a superspace $\tilde{X}$ of X and a clopen compact function $\tilde{f}$ from $\tilde{X}$ onto Y whose restriction to X is f. Moreover, the function $\tilde{f}$ is continuous, if the function f is continuous.

Throughout this chapter f will denote a function from a topological space X into another topological space Y; and the symbols f^*, f_1, X^*, q, W will have the same meaning as in the proof of the theorem 2.2, unless stated otherwise. Note that if the function f is open, then $X^* = X$ and $f^* = f$. Hence in this chapter, we assume that f is not open so that X^* is different from X.

Remark 2.4. The quotient mapping q restricted to X and each of Y_x is a homeomorphism. Furthermore, if X and Y are T_1-spaces, then the set $q(X)$ and each of the sets $q(Y_x)$ are closed subset of the space X^*. Hence if X and Y are T_1-spaces, then the family $\{q(X) \cup q(Y_x) : x \in S\}$ is a closed covering of the space X^*.

3. Co-reflexive and Connectedness properties of the space X^*

In this section we relate the co-reflexive and connectedness properties of the spaces X and Y with that of the space X^*. We show that any co-reflexive property of X and Y is carried over to X^*. Also we show that the (pathwise) connectedness of X and Y is carried over to X^*.

Definition 3.1. Let P be a topological property. We shall say that the space X^* preserves property P (or the property P is preserved in X^*) if X^* has property P whenever X and Y have property P. The property P is said to be a co-reflexive property if P is preserved under disjoint topological sums and quotients.

For a detailed study of co-reflective subcategories and co-reflexive properties see [30].

The following lemma is well known. We state it without proof.

Lemma 3.2. The property of being (i) a sequential space or (ii) a k-space or (iii) a c-space or (iv) a chain net space or (v) a locally (pathwise) connected space is a co-reflexive property.

The above lemma for (i) is due to Franklin (see [22]) and for (ii) see [59]. The lemma for (iii) was first proved by Moore and Mrówka (see [43]), and part (iv) by Herrlich (see [29]).

Proposition 3.3. Any co-reflexive property of X and Y is preserved in X^*. In particular, if X and Y are sequential spaces, k-spaces or c-spaces, so is X^*. If X and Y are locally (pathwise) connected or chain net spaces, so is X^*.

Proof. Since only disjoint topological sums and quotients were used in the construction of X^*, proposition follows immediately.

Since the property of being a sequential space or a k-space or a c-space or a chain net space is closed hereditary, we obtain the following corollary.

Corollary 3.4. Let X and Y be T_1-spaces. Then the space X^* is a sequential space (respectively, k-space c-space, chain net space) if and

only if X and Y are sequential spaces (respectively, k-spaces, c-spaces, chain net spaces).

Proposition 3.5. The space X^* is connected (respectively, pathwise connected, arcwise connected) whenever X and Y are connected (respectively, pathwise connected, arcwise connected).

Proof. If x is a singular point of X and $f(x) \in Y_x$, then $q(x) = q(f(x))$. If X and Y are connected (respectively, pathwise connected, arcwise connected) so are their homeomorphic images $q(X)$ and $q(Y_x)$. Since $q(X) \cap q(Y_x) \neq \phi$ for each $x \in S$, the union $\bigcup_{x \in S} (q(X) \cup q(Y_x)) = X^*$ is connected (respectively, pathwise connected, arcwise connected). The proof of the proposition is complete.

4. Separation properties of the space X^*.

In this section we relate the seperation properties of X, Y and X^* and show that the space X^* is a T_i-space if and only if X and Y are T_i-spaces where $i = 0, 1, 2, 3, 3\frac{1}{2}$.

Proposition 4.1. The space X^* is T_o if and only if the spaces X and Y are T_o.

Proof. Since X and Y are homeomorphically embedded in X^* and since T_o is a hereditary property, the spaces X and Y are T_o whenever X^* is. Conversely suppose that X and Y are T_o-spaces and let x_1, x_2 be any two distinct points in X^*. Consider $q^{-1}(x_1)$ and $q^{-1}(x_2)$. Essentially the following three cases arise.

Case I . The sets $q^{-1}(x_1)$ and $q^{-1}(x_2)$ are singletons. First suppose $q^{-1}(x_1)$ and $q^{-1}(x_2)$ lie in the same copy Y_x of Y. Since Y_x is a T_o-space there exists an open set N in Y_x containing one of the points

$q^{-1}(x_1)$ and $q^{-1}(x_2)$ which does not contain the other. Then the set $q(N \cup (W-Y_x))$ is a neighbourhood (in X*) of one of the points x_1 and x_2 which does not contain the other. Suppose $q^{-1}(x_1)$ and $q^{-1}(x_2)$ belong to two distinct copies Y_{x_i} and Y_{x_j} of Y. Since X is a T_0-space, there exists an open neighbourhood N in X of one of the points x_1 and x_2 which does not contain the other. Let us say that N is a neighbourhood of x_1. Then $q(N \cup (W-Y_{x_1}))$ is a neighbourhood (in X*) of x_1 which does not contain x_2. If one of $q^{-1}(x_1)$ and $q^{-1}(x_2)$ belongs to X and the other belongs to some copy Y_x of Y. Say $q^{-1}(x_1) \varepsilon X$ and $q^{-1}(x_2) \varepsilon Y_x$. Since X is a T_0-space there is an open neighbourhood N in X of one of the points $q^{-1}(x_1)$ and x which does not contain the other. Then $q(N \cup (W-Y_x))$ is a neighbourhood (in X*) of x_1 which misses x_2. If $q^{-1}(x_1)$ and $q^{-1}(x_2)$ both lie in X, then there is an open neighbourhood N in X of one of the points $q^{-1}(x_1)$ and $q^{-1}(x_2)$ which does not contain the other. Suppose N is a neighbourhood of $q^{-1}(x_1)$. Then $q(N \cup (W-Y_x))$ is a neighbourhood (in X*) of x_1 which does not contain x_2.

<u>Case</u> II. One of $q^{-1}(x_1)$ and $q^{-1}(x_2)$ is a singleton and the other is a doubleton. Suppose $q^{-1}(x_1) = \{x, f(x)\}$ and $q^{-1}(x_2)$ is a singleton. If $q^{-1}(x_2) \varepsilon X$, then there is an open neighbourhood N(in X) of one of the points x and $q^{-1}(x_2)$ that misses the other. Say N is a neighbourhood of x. Then $q(N \cup (W-Y_{q^{-1}(x_2)})$ is a neighbourhood (in X*) of x_1 which does not contain x_2. If $q^{-1}(x_2)$ belongs to some copy Y_{x_i} of Y, then there is a neighbourhood N(in X) of one of x and x_i which does not contain the other. Suppose N is a neighbourhood of x.

Then $q(N \cup (W-Y_{x_1}))$ is a neighbourhood (in X^*) of x_1 which does not contain x_2.

Case III. The sets $q^{-1}(x_1)$ and $q^{-1}(x_2)$ are doubletons. Suppose $q^{-1}(x_1) = \{x, f(x)\}$ and $q^{-1}(x_2) = \{y, f(y)\}$. Since X is a T_0-space, there exists a neighbourhood N (in X) of one of the points x and y which does not contain the other. Say, N is a neighbourhood of x. Then $q(N \cup (W-Y_y))$ is a neighbourhood (in X^*) of x_1 which does not contain x_2. This completes the proof of the proposition.

Proposition 4.2. The space X^* is T_1 if and only if the spaces X and Y are T_1.

Proof. The proof of necessity is immediate in view of the fact that the spaces X and Y are homeomorphically embedded in X^* and T_1 is a hereditary property. To prove sufficiency suppose that X and Y are T_1-spaces. Then W is a T_1-space. Let $p \in X^*$ be any point. Then $q^{-1}(p)$ is at most a doubleton and hence it is closed in W. This in turn implies that the singleton $\{p\}$ is closed in X^*. This completes the proof of the proposition.

Proposition 4.3. If X and Y are stratifiable spaces, then X^* is a stratifiable space.

Proof. By Proposition 4.2 the space X^* is a T_1-space. Since X and Y are T_1-spaces, the collection $\{q(X)\} \cup \{q(Y_x) : x \in S\}$ is a closed covering of the space X^* (see Remark 2.4). By Theorem 1.2.7(A), the space X^* is a stratifiable space.

Proposition 4.4. The space X^* is T_2 if and only if the spaces X and Y are T_2.

Proof. Necessity is obvious in view of the fact that the spaces X and Y are homeomorphically embedded in X^* and T_2 is a hereditary property. To prove sufficiency suppose X and Y are T_2 and let x_1, x_2 be any two distinct points in X^*. Consider $q^{-1}(x_1)$ and $q^{-1}(x_2)$. Essentially the following cases arise.

Case I. The sets $q^{-1}(x_1)$ and $q^{-1}(x_2)$ are singletons. Suppose $q^{-1}(x_1)$ and $q^{-1}(x_2)$ lie in the same copy Y_{x_i} of Y. Since the space Y_{x_i} is T_2, there are disjoint open neighbourhoods N_1, N_2 (in Y_{x_i}) of $q^{-1}(x_1)$ and $q^{-1}(x_2)$ respectively. Then $q(N_1)$ and $q(N_2)$ are disjoint neighbourhoods (in X^*) of x_1 and x_2 respectively. If $q^{-1}(x_1)$ and $q^{-1}(x_2)$ belong to two distinct copies Y_{x_i} and Y_{x_j} of Y respectively, then $q(Y_{x_i}-\{f(x_i)\})$ and $q(Y_{x_j}-\{f(x_j)\})$ are disjoint open neighbourhoods (in X^*) of x_1 and x_2 respectively. If one of $q^{-1}(x_1)$ and $q^{-1}(x_2)$ belongs to X and the other belongs to some copy Y_{x_i} of Y, say $q^{-1}(x_1) \in X$ and $q^{-1}(x_2) \in Y_{x_i}$, then $q^{-1}(x_1) \neq x_i$ and $q(W-Y_{x_i})$ and $q(Y_{x_i}-\{f(x_i)\})$ are disjoint open neighbourhoods (in X^*) of x_1 and x_2 respectively. If $q^{-1}(x_1)$ and $q^{-1}(x_2)$ both belong to X, then there are disjoint open neighbourhoods N_1 and N_2 (in X) of $q^{-1}(x_1)$ and $q^{-1}(x_2)$ respectively. For $i = 1,2$ let $\tilde{N}_i = N_i \cup (\cup Y_{x_k})$ where the second union is taken over all singular points x_k of X which are in N_i. Then $q(\tilde{N}_1)$ and $q(\tilde{N}_2)$ are disjoint open neighbourhoods (in X^*) of x_1 and x_2 respectively.

Case II. One of the sets $q^{-1}(x_1)$ and $q^{-1}(x_2)$ is a singleton and the other is a doubleton. Suppose $q^{-1}(x_1) = \{x, f(x)\}$ and $q^{-1}(x_2)$ is a singleton. If $q^{-1}(x_2) \in Y_{x_i}$ and $x = x_i$, then since Y_x is T_2, there exist disjoint open neighbourhood N_1 and N_2 (in Y_x) of $f(x)$ and $q^{-1}(x_2)$

respectively. The sets $q(N_1 \cup (W-Y_x))$ and $q(N_2)$ are disjoint neighbourhoods (in X*) of x_1 and x_2 respectively. If $x \neq x_i$, then $q(W-Y_{x_i})$ and $q(Y_{x_i} - \{f(x_i)\})$ are disjoint open neighbourhoods in (in X*) of x_1 and x_2 respectively. If $q^{-1}(x_2) \in X$, then there are disjoint open neighbourhoods N_1 and N_2 (in X) of x and $q^{-1}(x_2)$ respectively. For i =1, 2 let $\tilde{N}_i = N_i \cup (\bigcup Y_{x_\alpha})$ where the second union is taken over all singular points x_α of X which are in N_i. Then $q(\tilde{N}_1)$ and $q(\tilde{N}_2)$ are disjoint open neighbourhoods (in X*) of x_1 and x_2 respectively.

<u>Case</u> III. The sets $q^{-1}(x_1)$ and $q^{-1}(x_2)$ are doubletons. Say, $q^{-1}(x_1) = \{x, f(x)\}$ and $q^{-1}(x_2) = \{y, f(y)\}$. Since x and y are distinct and since X is a T_2-space, there are disjoint open neighbourhood N_1 and since X is a T_2-space, there are disjoint open neighbourhood N_1 and N_2 (in X) of x and y respectively. For i = 1, 2 let $\tilde{N}_i = N_i \cup (\bigcup Y_{x_k})$ where the second union is taken over all the singular points x_k of X which are in N_i. Then $q(\tilde{N}_1)$ and $q(\tilde{N}_2)$ are disjoint open neighbourhoods (in X*) of x_1 and x_2 respectively. This completes the proof of the Proposition.

The space X is said to be <u>Uryshon space</u> if for each pair of points $x, y \in X$ with $x \neq y$ there exist disjoint closed neighbourhoods U and V of x and Y respectively.

Let $x \in X$ be any point and let U be a closed neighbourhood of x. For each $x \in U \cap S$, let U_x denote a closed neighbourhood of $f(x)$ (in Y_x). Let $A = U \cup (\bigcup_x U_x)$ where the second union is taken over all x in $U \cap S$. Then $q(A)$ is a closed neighbourhood of $q(x)$ in X*. This fact together with a proof similar to that of proposition 4.4 yields the following result.

Proposition 4.5. *The space* X^* *is a Uryshon space if and only if* X *and* Y *are Uryshon spaces.*

Proposition 4.6. *The space* X^* *is* T_3 *if and only if* X *and* Y *are* T_3.

Proof. Since the property of being a T_3-space is hereditary, necessity is obvious. To prove sufficiency let F be a closed set in X^* and let $p \notin F$. If $q^{-1}(p) \cap X = \phi$, then $q^{-1}(p) \in Y_x$ for some x. Since Y_x is T_3, there are disjoint open neighbourhoods N_x and N_F (in Y_x) of $q^{-1}(p)$ and $(q^{-1}(F) \cap Y_x) \cup \{f(x)\}$(where $f(x) \in Y_x$) respectively. The sets $q(N_p)$ and $X^*-q(N_p)$ are disjoint neighbourhoods (in X^*) of p and F respectively. On the other hand, if $x_o \in q^{-1}(p) \cap X$, there are disjoint open neighbourhoods N_{x_o} and N_F (in X) of x_o and $q^{-1}(F) \cap X$ respectively (if $q^{-1}(F) \cap X = \phi$, let $N_F = \phi$). For each singular point $x \in S \cap N_{x_o}$, let N_x and N_{F_x} be disjoint open neighbourhoods (in Y_x) of $f(x)$ and $q^{-1}(F) \cap Y_x$ respectively (if $S \cap N_{x_o} = \phi$, let $N_x = \phi$). Let N denote the union of N_{x_o} and the sets N_x's obtained above. Then $q(N)$ and $W^*-q(N)$ are disjoint neighbourhoods (in X^*) of p and F respectively. This completes the proof of the proposition.

Proposition 4.7. *The space* X^* *is a Tychonoff space if and only if* X *and* Y *are Tychonoff spaces.*

Proof. The proof of necessity is immediate in view of the fact that X and are homeomorphically embedded in X^* and the property of being a Tychonoff pace is hereditary. To prove sufficiency suppose F is a closed subset f X^* and $p \notin F$. If $q^{-1}(p) \cap X = \phi$, then $q^{-1}(p) \in Y_x$ for some x. By omplete regularity of Y_x, there is a continuous real-valued function on Y_x which is zero at $q^{-1}(p)$ and one at $\{f(x)\} \cup (q^{-1}(F) \cap Y_x)$. xtend Φ continuously to all of W by constantly

taking one on X and on each $Y_{x'}$, $x' \neq x$. This extended Φ in turn defines a continuous real-valued function on X^* which separates p and F. In the other case, if $x_0 \in q^{-1}(p) \cap X$, let Φ_0 be a continuous real-valued function on X which is zero at x_0 and one on $q^{-1}(F) \cap X$. For each singular point of X, choose a continuous real-valued function Φ_x on Y_x which is one on $q^{-1}(F) \cap Y_x$ and $\Phi_x(f(x)) = \Phi_0(x)$. These functions Φ_x together with Φ_0 combine to form a continuous real-valued function on W which in turn induces a continuous real-valued function on X^* separating p and F. This completes the proof of the proposition.

A Tychonoff space X is said to be a P-space if each G_δ set in X is open.

Since the property of being a P-space is hereditary and since completely regular quotient of a P-space is a P-space (see [25], Problem 4K) ,by proposition 4.7 we have

Proposition 4.8. The space X^* is a P-space if and only if X and Y are P-spaces.

Proposition 4.9. The space X^* is functionally Hausdorff if and only if the spaces X and Y are functionally Hausdorff.

Proof of proposition 4.9 is same as that of the proposition 4.7 with closed set replaced by a point and the role of complete regularity replaced by functional Hausdorffness.

5. The Adjunction space representation, Normality Properties and Paracompactness of the space X^*

In this section we show that if X and Y are T_1-spaces, then the space X^* can also be obtained as an adjunction space. Using this

representation of X^* we relate the normality properties and paracompactness of X, Y with that of the space X^*. It turns out that the space X^* is paracompactness (respectively normal) if and only if X and Y are paracompact (respectively normal).

The Adjunction space representation 5.1. Let F be the subset of $\bigoplus_{x\varepsilon S} Y_x$ whose intersection with each Y_x is its point $f(x)$. Since Y is a T_1-space, the set F is a closed and discrete subset of $\bigoplus_{x\varepsilon S} Y_x$. The mapping g from F into X which sends each $f(x)$ to x, yields the adjunction space $(\bigoplus_{x\varepsilon S} Y_x) \cup_g X$ which is homeomorphic to X^*.

Before proving the next proposition we state a lemma without proof which will be of use later on.

Lemma 5.2. Let $\{X_\alpha\}_{\alpha\varepsilon I}$ be a family of topological spaces such that each X_α has property P where P is any one of the following properties

(1) normality, (2) hereditary normality, (3) perfect normality,

(4) collectionwise normality (5) full normality (=paracompactness)

(6) m-paracompactness and normality, where m is any infinite cardinal

(7) hereditary paracompactness (8) hereditary collectionwise normality

(9) hereditary m-paracompactness and normality (10) completeness in the sense of Čech.

Then the disjoint topological sum $\bigoplus_{\alpha\varepsilon I} X_\alpha$ also has the property P.

For a proof of the lemma for properties (1), (5) and (10) see ([20], pages 71, 218 and 144).

From here onward all the spaces are assumed to be T_1, unless expressly stated otherwise.

Proposition 5.3. If P is any one of the properties (1) - (9) of lemma 5.2 then the space X^* has property P if and only if X and Y have the property P.

Proof. Necessity is immediate in view of the fact that X and Y are homeomorphically embedded in X^* as closed subspaces (see remark 2.4) and the property P is closed hereditary. The proof of sufficiency follows from the fact that the space X^* can be obtained as an adjunction space of the spaces $\bigoplus_{x \in S} Y_x$ and X via the map g and that the property P is preserved under disjoint topological sums and adjunctions (see Theorems I.2.2 and I.2.3).

6. Properties not preserved in X^*

In this section we show that many topological properties are not necessarily preserved in X^*. The pertinent example which shows that X^* need not preserve many topological properties is as follows :

Example 6.1. Let X be the plane set consisting of the union of the closed intervals $[-1,1]$ on the two axes. Let $Y = [-1, 1]$ on the x-axis and let f be the restriction to X of the usual projection on the x-axis. Then each point on the y-axis, except the origin is a singular point of X. Construct X^* as in the theorem 2.2. Then no point in X^* which is the image of a point $(0,y) \in X$ under q has a countable base of neighbourhoods. Hence X^* is not first countable and consequently X^* is neither second countable nor metrizable. Let $\{x_n\}_{n \in \mathbb{N}}$ be a sequence in X^* where $x_n = q(\frac{1}{2}) \in q(Y_{(0,\frac{1}{n})})$. Then the sequence $\{x_n\}_{n \in \mathbb{N}}$ has no cluster point. Hence X^* is not countably compact and therefore X^* is neither sequentially compact nor compact.

Since X^* is normal (see proposition 5.3) and since every normal pseudocompact space is countably compact ([25], Problem 3D.2), the space X^* is not pseudocompact. Let $x = (0,0)$ and let N be a neighbourhood in X^* of $q(x)$. Let $\{y_n\}_{n\in\mathbb{N}}$ be an infinite sequence in S (the set of singular points of X) such that $N \cap q(Y_{y_n}) \neq \phi$ for each n. Choose $z_n \in N \cap q(Y_{y_n} - \{f(y_n)\})$. Then $\{z_n : n\in \mathbb{N}\}$ is an infinite discrete subset of N which has no limit point. This implies N is not compact. Therefore X^* is not locally compact. The space X^* is neither separable nor Lindelöf. Hence we have the following proposition.

Proposition 6.2. The space X^* need not preserve metrizability, either axioms of countability, weight or local weight, (hereditary) separability, the (hereditary) Lindelöf property, compactness, countable compactness or sequential compactness or pseudocompactness or local compactness.

Definition 6.3. Let P be a topological property and let X denote any nonempty set. Let $P(X)$ denote the set of topologies on X with property P. Note that $P(X)$ is partially ordered by inclusion. A topological space (X,T) is said to be minimal P (or P-minimal) provided T is a minimal element in $P(X)$. We say that (X,T) is P-closed if (X,T) is a closed subspace of each of its superspaces Y with property P.

Let P represent either the Hausdorff, Uryshon, regular, completely regular, or normal separation axioms or paracompactness. Since the space X^* in example 6.1 is a non-compact paracompact Hausdorff space, it is not P-closed (see [7], Theorems 3.3(a), 4.3(d), 4.16(e) and 5.1). Since a P-minimal space is P-closed (see [7], Theorems 3.1(a), 4.3(a), 4.16(a) and 5.1), the space X^* in example 6.1 is not P-minimal.

If P represents the axiom of perfect normality, then the notions of P-minimal, P-closed and countable compactness coincide (see [7], Theorem 5.2). Since the space X* in Example 6.1 is a perfectly normal but not countably compact space, it is not minimal perfectly normal. Thus we have the following proposition.

Proposition 6.4. Let P represent the Hausdorff, Uryshon, regular or completely regular separation axioms or paracompactness, metrizability, local compactness or perfect normality. Then X* need not preserve the property of being P-minimal or P-closed.

As shown above many topological properties are not preserved in X*. Let P be a topological property which is not necessarily preserved in X*. It seems natural to ask the question : Under what conditions on the mapping f and or additional hypothesis on spaces X and Y, the property P is preserved in X* ? We have succeeded in answering this question for many topological properties. In preceding sections we answer this question to some extent by showing that the space X* may be induced to preserve many topological properties by imposing restrictions on the set S of singular points of X.

7. Compactness Properties of the space X*

The proposition 5.3 tells us that paracompactness is preserved in X*. The following proposition acquaint us with the situation for many other forms of compactness.

Proposition 7.1. Let P be any one of the following properties.

(1) compactness (2) countable compactness (3) sequential compactness

Then the space X^* has property P if and only if X and Y have property P and the set S of singular points of X is finite.

Proof. The proof of sufficiency is obvious in view of the fact that properties (1) - (3) are preserved under finite disjoint topological sums and continuous surjections. To prove necessity suppose X^* has property P. Since each of the properties (1) - (3) is closed hereditary, the spaces X and Y have the property P. Now it suffices to show that S is finite. Assume the contrapositive. Since Y is non-degenerate, choose for each singular point x of X, a point $y_x \varepsilon q(Y_x - \{f(x)\})$. Then the set $\{y_x : x \varepsilon S\}$ is an infinite subset of X^* which has no limit point. This is a contradiction to the fact that X^* has property P. This completes the proof of the proposition.

The forthcoming theorem partially answers the question for preservation of pseudocompactness in X^*.

Proposition 7.2. If the set S of singular points of X is finite then X^* preserves pseudocompactness. Conversely if X^* is pseudocompact then X is pseudocompact and in addition if f is continuous then Y is also pseudocompact. Further if Y is functionally Hausdorff, then the set S is finite.

Proof. Since pseudocompactness is preserved under finite disjoint topological sums and continuous surjections, the space X^* is pseudocompact whenever X and Y are. Conversely suppose X^* is pseudocompact and let $g \varepsilon C(X)$ be any function. Then for each singular point x of X, let Φ_x be the constant function on Y_x which constantly takes the value $g(x)$. Then g together with the functions Φ_x's combine to form a continuous real valued function on W. This in turn induces a continuous real-valued

function G on X^* whose restriction to X is the function g. Consequently g is bounded. Continuity of the function f implies that of f^* (Theorem 2.2). Therefore, Y being a continuous image of X^* is pseudocompact. Let Y be functionally Hausdorff and assume that S is infinite. Let $S' = \{x_n : x_n \in S, n \in \mathbb{N}\}$ be a countably infinite subset of S and let g be any continuous function from X into the unit interval $[0,1]$. For each $x \in S-S'$ let g_x be the constant function on Y_x which constantly takes the value $g(x)$. For each $x_n \in S'$, let g_{x_n} be a continuous function from Y_{x_n} into the interval $[g(x_n),n]$ such that $g_n(f(x_n)) = g(x_n)$ and g_n takes the value n at least at one point. Then these g_n's together with the function g combine to form an unbounded continuous real-valued function G on W. This function G in turn induces an unbounded continuous real-valued function on X^*. This contradicts the assumption that X^* is pseudocompact. Therefore the set S of singular points of X is finite. The proof of the proposition is complete.

The next proposition tells how the separability and the Lindelöf properties of the spaces X, Y and X^* are related.

Proposition 7.3. Let m be an infinite cardinal number and let P be any one of the following properties.

(1) m-separability (2) m-Lindelöf property (3) hereditary m-separability (4) hereditary m-Lindelöf property.

Then the space X^* has property P if and only if X and Y have property P and the set S of singular points of X is of cardinality at most m.

We omit the proof of proposition 7.3.

We shall say that singular points of X do not accumulate in case S is a closed subset of X and is discrete in its relative topology from X.

The following result partially answers the question for preservation of metacompactness in X^*.

Proposition 7.4. *Let X and Y be T_1-spaces. If the spaces X and Y are metacompact and if the singular points of X do not accumulate, then the space X^* is metacompact.*

Proof. Since the singular points of X do not accumulate, the closed covering $\{q(X) \cup q(Y_x) : x \in S\}$ is a locally finite collection. Since the spaces X and Y are metacompact, $q(X) \cup q(Y_x)$ is metacompact for each $x \in S$. The proposition follows at once from Theorem I.7.3.

We do not know whether realcompactness is preserved in X^* or not. Mrowka in [47] showed that if a normal space X is the union of a countable collection of closed realcompact spaces, then X is realcompact. Hence the following holds true.

Proposition 7.5. *If X and Y are normal realcompact spaces and if the set S of singular points of X is countable, then the space X^* is realcompact.*

For preservation of realcompactness in X^* also see Proposition 8.16.

8. Metrizability, weight, local weight and local compactness of the space X^*.

In section 6 of this chapter it was shown that X^* need not preserve metrizability, weight, local weight and local compactness. Here we give necessary and sufficient conditions for the preservation of

metrizability, axioms of countability and local compactness in X^*. Also we give sufficient conditions for the preservation of completeness in the sense of Čech, strong paracompactness and weight (or local weight) strictly larger than $\aleph_0$.

The following lemma will be useful in proving many results in the sequel.

Lemma 8.1. If the singular points of X do not accumulate then the quotient map q from W onto X^* is a perfect map.

Proof. We first show that the quotient map q is a closed map. Suppose F is a closed subset of W. Let $\{p_\alpha\}_{\alpha\varepsilon D}$ be a net in $q(F)$ converging to a point p in X^*. Now the set $q(F) = q(F \cap X) \cup (\bigcup_{x \varepsilon S} q(F \cap Y_x))$. Since the restriction of q to X and each Y_x is a homeomorphism and since the set $q(X)$ and each of the sets $q(Y_x)$ are closed in X^*, the set $q(F \cap X)$ and each of the sets $q(F \cap Y_x)$ are closed in X^*. Thus if the net $\{p_\alpha\}_{\alpha\varepsilon D}$ is frequently in $q(F \cap X)$ or in some $q(F \cap Y_x)$, then the point $p \varepsilon q(F)$ and we are done. If $q^{-1}(p) = \{y\}$ with $y \varepsilon Y_x$ let $U = Y_x$. If $q^{-1}(p) = \{x\}$, where x is a non-singular point of X, let U be a neighbourhood of x in X which is free of singular points. If $q^{-1}(p) = \{x, f(x)\}$ let U denote the union of a neighbourhood of x in X which contains no singular point of X other than x and the set Y_x. In any case $q(U)$ is a neighbourhood of p which has a nonempty intersection with at most one $q(Y_x \cap F)$. But the net $\{p_\alpha\}_{\alpha\varepsilon D}$ is eventually in $q(U)$ and hence is frequently in either $q(U) \cap q(F \cap X)$ or $q(U) \cap q(F \cap Y_x)$ for some x. Hence q is a closed map. Since each $q^{-1}(p)$ is at most a doubleton, the map q is a perfect map. The proof of the lemma is complete.

The following lemma is a particular case of a lemma due to Henriksen and Isbell (see [28], lemma 1.5) and will be useful in proving the next forthcoming lemma.

Lemma 8.2. *If f is a continuous mapping of a Tychonoff space X onto a Tychonoff space Y, then the following statements are equivalent.*

(1) *The map f is a perfect map.*

(2) *The image of $\beta X-X$ under the extension map f^{β} (of f to its Stone-Čech compactification) is precisely $\beta Y-Y$.*

The following lemma seems to be well known.

Lemma 8.3. *Let f be a perfect map from a topological space X onto a topological space Y and let P be any one of the following properties. (1) metrizability (2) local compactness (3) weight m, where m is an infinite cardinal number (4) completeness in the sense of Čech.*

Then Y has property P whenever X has property P.

The proof for property (1) was first given by Hanai and Dugundji gives a quick proof of the lemma for properties (1) and (2) (see [19], pages 235, 240). The proof for the property (3) in case $m = \aleph_0$ is also given in [19]. Though the proof is similar in the general case we give it here merely for the sake of completeness. Let the weight of X be m and let $\{U_\alpha\}_{\alpha\epsilon I}$ be a base of X of the cardinality m. Let $\{V_\beta\}_{\beta\epsilon J}$ be the family of all finite unions of U_α. Then the family $\{V_\beta\}_{\beta\epsilon J}$ is of cardinality no larger than m. We shall show that the open sets $W_\beta = Y-f(X-V_\beta)$ form a base for Y. Let $y \epsilon Y$ and let W be an open set containing y. Then $f^{-1}(y) \subset f^{-1}(W)$ and since $f^{-1}(y)$ is compact, there are finitely many sets $U_{\beta_1}, \ldots, U_{\beta_n}$ such that $f^{-1}(y) \subset \bigcup_{i=1}^{n} U_{\beta_i} = V_\beta \subset f^{-1}(W)$.

Then $y \in W_\beta \subset f(V_\beta) \subsetneq W$ and this completes the proof for property (3).

To prove the lemma for property (4), suppose X is complete in the sense of Čech. Then there is a sequence $\{F_i\}_{i \in N}$ of closed sets in βX such that $\beta X - X = \bigcup_{i=1}^{\infty} F_i$. By lemma 8.2 we have that $f^\beta(\beta X - X) = \bigcup_{i=1}^{\infty} f^\beta(F_i) = \beta Y - Y$ Since each of the sets $f^\beta(F_i)$ is closed in βY, the space Y is complete in the sense of Čech. This completes the proof of the lemma.

Proposition 8.4. Suppose the singular points of X do not accumulate and suppose P is any one of the following properties.
(1) metrizability (2) local compactness (3) local weight m (4) completeness in the sense of Čech (5) strong paracompactness.

Then X^* preserves the property P. Further if the cardinality of the set S of singular points of X is no larger than the maximum of the weights of X and Y, then the weight is also preserved in X^*.

Proof. Let P be any one of the properties (1), (2) and (4). Since the property P is preserved under disjoint topological sums (see lemma 5.2) and perfect maps (see lemma 8.3), by lemma 8.1 the space X^* has property P. To prove the proposition for property (3), suppose m is the larger of the local weights of X and Y. Let p be any point of X^*. If $q^{-1}(p) = \{y\}$ where $y \in Y_X - \{f(x)\}$, the image of a base at y under q is the base at p. If $q^{-1}(p) = \{x\}$ where x is a nonsingular point of X, a base at x can be chosen of cardinality at most m whose members contain no singular points of X. The image of this base under q is a base at p. If $q^{-1}(p) = \{x, f(x)\}$, choose a base B at x whose members contain only one singular point, and choose a base U at $f(x) \in Y_X$ with cardinality of U and B

no larger than m. The images under q of the sets of the form $B \cup V$ with $B \in \mathcal{B}$ and $V \in \mathcal{V}$, form a base at p of cardinality no larger than m.

To prove the proposition for property (5), let $\{U_\alpha\}_{\alpha\in I}$ be an open cover of X^*. Then $\zeta_X = \{q^{-1}(U_\alpha) \cap Y_X\}_{\alpha\in I}$ is an open cover of Y_X and $\zeta_X = \{q^{-1}(U_\alpha) \cap N_x : x \in X$ and N_x is an open neighbourhood of x which contains no singular point of X (possibly other than x), $\alpha \in I\}$ is an open cover of X. Let ζ^X be an open star finite refinement of ζ_X and let ζ^X be an open star finite refinement of ζ_X. Let $\zeta = \zeta^X \cup (\bigcup_{x\in S} \zeta^x)$. For each $U \in \zeta$, define

$$\tilde{U} = \begin{cases} U \text{ if } U \text{ contains no singular point of either } X \text{ or } Y. \\ U \cup U_{f(x)} \text{ where } f(x) \in U_{f(x)} \in \zeta^X \text{ and } U \cup U_{f(x)} \subseteq q^{-1}(U_\alpha) \\ \qquad \text{for some } \alpha, \text{ if } U \text{ contains a singular point } x \text{ of } X. \\ U \cup U_x \text{ where } x \in U_x \in \zeta^X \text{ and } U \cup U_x \subseteq q^{-1}(U_\alpha) \text{ for some} \\ \qquad \alpha, \text{ if } f(x) \in U \in \zeta^X \end{cases}$$

Then the family $\tilde{\zeta} = \{q(\tilde{U}) : U \in \zeta\}$ is an open star finite refinement of the covering $\{U_\alpha\}_{\alpha\in I}$.

To prove the last assertion of the proposition we need only note that weight W = weight $X + |S|$. weight $Y \leq m + m^2 = m$ and that perfect maps do not raise weight (Lemma 8.3).

The example 6.1 shows that the restrictions imposed on the set S of singular points of X in the proposition 8.4 are not superfluous. By replacing the interval $[-1,1]$ on the y-axis in example 6.1 by a

sequence converging to zero one can easily see that these cannot be weakened even to S being a countable discrete set for the properties (1), (2), (3) and the property of being second countable.

Having this state of affairs one is naturally led to ask : Are the restrictions on the set S of singular points of X in the proposition 8.4 necessary for the properties (1) - (3) ? The following discussion will answer this question completely.

First we quote an example due to Franklin [22] which will be referred to later on.

Example (Franklin) 8.5. A sequential space which is not Fréchet and hence not first countable. Let $M = (\mathbb{N} \times \mathbb{N}) \cup \mathbb{N} \cup \{0\}$ with $(m,n) \in \mathbb{N} \times \mathbb{N}$ an isolated point, where $\mathbb{N}$ denotes the set of natural numbers. For a basis of neighbourhoods at $n_o \in \mathbb{N}$, take all sets of the form $\{n_o\} \cup \{(m,n_o) : m \geq m_o\}$. A set U will be a neighbourhood of 0 if and only if $0 \in U$ and U is a neighbourhood of all but finitely many $n \in \mathbb{N}$. Then M is a sequential space and contains a nonsequential subspace. Therefore M is not a Fréchet space and hence a non-first countable space.

Suppose singular points do accumulate and let x be a limit point of the set S. If X is metrizable (or first countable) there is a sequence $\{x_n\}_{n\in\mathbb{N}}$ in S which converges to x. Since each x_i is a singular point, the point $f(x_i)$ is a non-isolated point of Y. If Y is metrizable (or first countable), there is a sequence $\{y_{ij}\}_{j\in\mathbb{N}}$ of distinct points in Y which converges to $f(x_i)$. Choose this sequence $\{y_{ij}\}_{j\in\mathbb{N}}$ from Y_{x_i} which converges to $f(x_i) \in Y_{x_i}$. Now the space $Z = q(\{x\} \cup \{x_n\}_{n\in\mathbb{N}} \cup (\bigcup_i \{y_{ij}\}_{j\in\mathbb{N}}))$ can be easily seen to be

homeomorphic to the space M of the Example 8.5. Therefore X^* is not first countable and hence neither second countable nor metrizable. Combining this fact together with the Proposition 8.4 we have the following results.

Theorem 8.6. The space X^* is metrizable (respectively, first countable) if and only if X and Y are metrizable (respectively, first countable) and singular points of X do not accumulate.

Theorem 8.7. The space X^* is second countable if and only if X and Y are second countable and the set S of singular points of X is countable and has no limit point.

Proposition 8.8. The space X^* is Fréchet if and only if X and Y are Fréchet and singular points of X do not accumulate.

A topological space X is said to be completely metrizable if there exists a complete metric compatible with the topology of the space.

Čech proved the following result for completely metrizable spaces (see [52] , page 205) which we shall use in proving the next result. We state it as a lemma.

Lemma (Čech) 8.9. A metrizable space X is completely metrizable if and only if it is complete in the sense of Čech.

Theorem 8.10. The space X^* is completely metrizable if and only if X and Y are completely metrizable and singular points of X do not accumulate.

Proof. Suppose X^* is completely metrizable. Since X and Y are homeomorphically embedded in X^* as closed subspaces and since the property of complete metrizability is closed hereditary, the spaces X and Y are completely metrizable. By metrizability of X^* it follows that the singular points of X do not accumulate (see Theorem 8.6).

The proof of sufficiency follows from the Proposition 8.4 and lemma 8.9.

In analogy to Theorem 8.6 and 8.7 one may as well suspect that the conditions on the set S of singular points of X in the Proposition 8.4 are also necessary for the preservation of weight (or local weight) strictly larger than $\aleph_0$. This is not true as shown by the following example.

Example 8.11. Let $X = [0,1]$ be such that every point of X is discrete except the point 0. A set N is a neighbourhood of 0 if and only if $0 \in N$ and N contains all but finitely many points of the sequence $\{\frac{1}{n}\}_{n\in \mathbb{N}}$. Let $Y = [0,1]$ with the cofinite topology and let f be the identity map of X onto Y. Then every point of X is a singular point of X and 0 is a limit point of the set X. Local weight (respectively, weight) $X = \aleph_0$ (respectively, c) and local weight (respectively, weight) $Y = c$ (respectively, c). Then it is routine to verify that local weight (respectively, weight) $X^* = c$(respectively, c).

The above example has been constructed when the larger of the weights (local weights) of X and Y equals c but the same technique can be employed to construct examples for any cardinal number $\aleph_\alpha > \aleph_0$.

In proving the next two propositions we make use of the fact that $\aleph_\alpha^{\aleph_{\alpha-1}} = \aleph_\alpha$ for $\alpha \geq 1$. The proof of this fact seems to make use of the generalized continuum hypothesis.

Proposition 8.12. If the maximum of the local weights of X and Y is $\aleph_\alpha > \aleph_0$ and if each limit point of the set S of singular points of X has a neighbourhood which contains at most $\aleph_{\alpha-1}$ points of S, then local weight is preserved in X^*.

Proof. Let $p \in X^*$ be any point. If $p \in q(Y_x)-\{q(x)\}$ for some $x \in S$, then $q^{-1}(p) = \{y\}$ where $y \in Y_x-\{f(x)\}$. The image of a base at y (in Y_x) under q is a base at p. If $p \in q(X)$, then $p = q(x)$ for some $x \in X$ choose a base $\mathcal{B}_x$ at x of cardinality no larger than $\aleph_\alpha$ such that each member of $\mathcal{B}_x$ contains at most $\aleph_{\alpha-1}$ points of the set S. For each $B \in \mathcal{B}_x$ and for each $x \in B \cap S$ choose a base $\mathcal{B}_{f(x)}$ at $f(x) \in Y_x$ of cardinality at most $\aleph_\alpha$. The images under q of the sets of the form $B \cup (\bigcup_{x\in B \cap S} B_{f(x)})$ with $B \in \mathcal{B}_x$ and $B_{f(x)} \in \mathcal{B}_{f(x)}$ form a base at p of cardinality at most $\aleph_\alpha$. The proof of the proposition is complete.

Proposition 8.13. <u>Let the maximum of the weights of X and Y be $\aleph_\alpha > \aleph_0$. If each limit point of the set S of singular points of X has a neighbourhood which contains at most $\aleph_{\alpha-1}$ points of S and if the cardinality of the set S is no larger than $\aleph_\alpha$, then weight is preserved in X*.</u>

Proof. Choose a base $\mathcal{B}$ of X such that each member of $\mathcal{B}$ contains at most $\aleph_{\alpha-1}$ points of the set S and such that the cardinality of $\mathcal{B}$ is no larger than $\aleph_\alpha$. For each $x \in S$, choose a base $\mathcal{B}_x$ of Y_x of cardinality no larger than $\aleph_\alpha$. Let

$$U = \begin{cases} B & \text{if } B \in \mathcal{B} \text{ and } B \text{ contains no singular point of } X. \\ B_x & \text{if } B_x \in \mathcal{B}_x \text{ and } f(x) \in B_x \\ B \cup (\bigcup_x B_x) & \text{where the second union is taken over the} \\ & \text{set } B \cap S \text{ and } B_x \text{ is a neighbourhood of } f(x) \\ & \text{if } B \in \mathcal{B} \text{ and } B \cap S \neq \phi \end{cases}$$

Then the image under q of the sets of the form U is a base for X* of cardinality no larger than $\aleph_\alpha$. The proof of the proposition is complete.

The hypothesis on the set S of singular points of X in the propositions 8.12 and 8.13 cannot be dropped as is suggested by the following example.

Example 8.14. Let $X = [0,1]$ with the usual topology and let Y be the space with the same set and cofinite topology. Let f be the identity map of X onto Y. Then every point of X is a singular point. Construct the space X^* and the mapping f^*. Then it is easily varified that weight (local weight) $X^* = 2^c > c$.

The following result shows that the hypothesis on the set S of singular points of X in proposition 8.4 is necessary for preservation of local compactness in X^*.

Proposition 8.15. The space X^* is locally compact if and only if X and Y are locally compact and singular points of X do not accumulate.

Proof. Sufficiency is proved in Proposition 8.4. To prove necessity suppose X^* is locally compact. Since X and Y are embedded in X^* as closed subspaces and since local compactness is closed hereditary, the spaces X and Y are locally compact. To show that singular points of X do not accumulate assume contrapositive. Let x be a limit point of the set S and let K be a compact neighbourhood (in X^*) of $q(x)$. Then $K \cap q(Y_{x_\alpha} - \{f(x_\alpha)\}) \neq \phi$ for infinitely many $x_\alpha \in S$. Let $A = \{y_\alpha : y_\alpha \in K \cap q(Y_{x_\alpha} - \{f(x_\alpha)\}) \neq \phi,\ x_\alpha \in S\}$. Then A is an infinite discrete subset of K. Since $q(Y_x - \{f(x_\alpha), y_\alpha\})$ is open for each $x_\alpha \in S$, the set $K \cap \bigcap_{x \in S} (q(Y_{x_\alpha}) - \{y_\alpha\})$ is a neighbourhood of $q(x)$ which does not intersect A. Thus A has no limit point. A contradiction to the fact that K is compact. Therefore singular points of X do not accumulate. The proof of the proposition is complete.

Mrówka in [47] showed that a nonmeasurable disjoint topological sum of realcompact spaces is realcompact. Frolík has shown that perfect image of a normal countably paracompact and realcompact space is realcompact (see [24], Theorem 12). Using these results together with Lemma 8.1 and Proposition 5.3 we obtain the following partial result about realcompactness of X^*.

Proposition 8.16. *Let X, Y be normal, countably paracompact and realcompact spaces. If the singular points of X do not accumulate and if the cardinal number $|S|$ is non-measurable, then the space X^* is realcompact.*

9. Dimension of the space X^*.

In this section we make an attempt to relate the dimension of the spaces X and Y with the dimension of the space X^*. It is shown that for normal spaces the covering dimension of X^* is equal to the larger of the covering dimensions of X and Y. Also it is shown that for perfectly normal spaces large inductive dimension is preserved in X^*.

First we prove the following lemma which will be of use in the sequel.

Lemma 9.1. *If X and Y are T_1-spaces, then X^* has the weak topology with respect to the closed covering* $\{q(X) \cup q(Y_x) : x \in S\}$.

Proof. Let B be any subset of S. Let $\bigcup_{x \in B} (q(X) \cup q(Y_x)) = A$ and let F be a subset of A such that $F \cap (q(X) \cup q(Y_x))$ is closed in $q(X) \cup q(Y_x)$ for each $x \in B$. Since the complement $X-A$ is the union $\bigcup_{x \in S-B} (q(Y_x)-\{q(x)\})$ and since each of the sets $q(Y_x)-\{q(x)\}$ is open in X^*,

the set A is closed in X^*. Since each of the sets $q(X) \cup q(Y_x)$ is closed in X^*, the set $F \cap (q(X) \cup q(Y_x))$ is closed in X^*. Thus the set $q^{-1}(F \cap (q(X) \cup q(Y_x)))$ is closed in W. Since for each $x \in B$ the set $q^{-1}(F) \cap X = q^{-1}(F \cap (q(X) \cup q(Y_x)))$ and the set $q^{-1}(F) \cap Y_x = q^{-1}(F \cap (q(X) \cup q(Y_x)))$, and since for each $x \in S-B$ the set $q^{-1}(F) \cap Y_x$ is either empty or at most a singleton, the set $q^{-1}(F)$ is closed in W. Consequently F is closed in X^*. The proof of the lemma is complete.

Proposition 9.2. *If X and Y are T_4-spaces, then $\dim X^*$ is equal to the maximum of $\dim X$ and $\dim Y$.*

Proof. Since X and Y are T_4-spaces, by Proposition 4.7 the space X^* is a T_4-space. Since X and Y are homeomorphically embedded in X^* as closed subspaces, $\dim X^*$ is not less than the maximum of $\dim X$ and $\dim Y$ (see, Theorem I.5.5). Since X^* is a T_4-space and since X^* has the weak topology with respect to the closed covering $\{q(X) \cup q(Y_x) : x \in S\}$, it follows from Morita's result (Theorem I.6.1) that $\dim X \leq \dim (q(X) \cup q(Y_x))$. Note that the space $q(X) \cup q(Y_x)$ has the weak topology with respect to the closed covering $\{q(X), q(Y_x)\}$. Hence by Theorem I.6.1 $\dim (q(X) \cup q(Y_x)) \leq \max \{\dim X, \dim Y\}$. Therefore $\dim X^* = \max \{\dim X, \dim Y\}$.

Since the conditions $\operatorname{Ind} X = 0$ and $\dim X = 0$ are equivalent for every T_4-space, we have the following corollary.

Corollary 9.3. *If X and Y are T_4-spaces and $\operatorname{Ind} X = 0 = \operatorname{Ind} Y$, then $\operatorname{Ind} X^* = 0$.*

Since the conditions $\operatorname{ind} X = 0$, $\dim X = 0$ and $\operatorname{Ind} X = 0$ are equivalent for every Lindelöf space, we obtain the following Corollary.

Corollary 9.4. *Let X and Y be regular Lindelöf spaces. Let the set S of singular points of X be countable and let $\operatorname{Ind} X = 0 = \operatorname{Ind} Y$.*

Then Ind $X^* = 0 =$ ind X^*.

Proposition 9.5. If X and Y are perfectly normal spaces, then large inductive dimension of X^* is equal to the maximum of the large inductive dimensions of X and Y.

Proof. Let n be the maximum of Ind X and Ind Y. Since X and Y are homeomorphically embedded in X^* as closed subspaces, Ind $X^* \geq n$. Since X and Y are perfectly normal spaces, by Proposition 5.3 the space X^* is perfectly normal. The space $X^*-q(X)$ is homeomorphic to the space $\bigoplus_{x\in S} (Y_x-\{f(x)\})$. By ([20], Problem 7.N) Ind $Y_x-\{f(x)\} \leq$ Ind $Y \leq n$. Thus Ind $(X^*-q(X)) \leq n$. By ([51], Theorem B, page 200) Ind $X^* \leq n$. Hence Ind $X^* = n$. The proof of the proposition is complete.

Lemma 9.1, though simple, is a useful tool in the investigation of the properties of the space X^*. In addition to the Proposition 9.2, many other results of the preceding sections can be easily proved with its help. For example, the Proposition 5.3 for properties (i) - (vi) follows in the light of Lemma 9.1 together with Theorems I.7.1 and I.7.2.

10. Mapping properties of the function f^*

The purpose of this section is to relate the mapping properties of the function f and f^*. We show that f^* is a connected function (respectively, connectivity function) if and only if f is a connected function (respectively, connectivity function). We also show that if the set S of singular points of X is finite, then the function f^* is closed if and only if the function f is closed.

Definition 10.1. Let f be a function from a topological space X into a topological space Y. Then graph of f is the set $\{(x,f(x)): x \in X\} \subset X \times Y$ and is denoted by '$G(f)$'. We say that f is connected function if $f(A)$

is connected for every connected set A in X. The function f is said to be a <u>connectivity function</u> if $G(f/A)$ is a connected set for every connected set A in X.

Every continuous function is a connectivity function and every connectivity function is a connected function but the converse is not true.

<u>Proposition</u> 10.2. <u>The function f^* is connected if and only if f is a connected function.</u>

<u>Proof.</u> Since the restriction of a connected function is a connected function, necessity follows. In order to prove sufficiency suppose that the function f is connected and let A be a connected subset of X^*. Essentially the following two cases arise.

<u>Case</u> I. Either $A \subset q(X)$ or $A \subset q(Y_x)$ for some $x \in S$. Since the restriction $f^*/q(X)$ is topologically equivalent to f and since the restriction $f^*/q(Y_x)$ is topologically equivalent to the identity map 1_Y, in either case $f^*(A)$ is a connected set.

<u>Case</u> II. Neither $A \not\subset q(X)$ nor $A \not\subset q(Y_x)$ for each $x \in S$. Then $A \cap q(X) \neq \phi$ and $A \cap q(Y_x) \neq \phi$ for at least one $x \in S$. Let S' denote the set of all points $x \in S$ such that $A \cap q(Y_x) \neq \phi$. We claim that the sets $A \cap q(X)$ and $A \cap q(Y_x)$ are connected. For if $A \cap q(X)$ is not connected, then there exists a partition $A \cap q(X) = A_1 \cup A_2$ of $A \cap q(X)$ Let $\tilde{A}_i$ denote the union of A_i and all $A_i \cap q(Y_x)$ such that $q(x) \in A_i$ for $i = 1, 2$. Then $A = \tilde{A}_1 \cup \tilde{A}_2$ is a partition of A. Similarly, if $A \cap q(Y_x)$ is not connected, then a partition of $A \cap q(Y_x)$ will induce a partition of A. If X and Y are T_1-spaces, then $X^* - q(Y_x)$ and

$q(Y_x)-\{q(x)\}$ are open in X^*. We claim that if $x \in S'$, then $q(x) \in A$. For if $q(x) \notin A$, then $A \cap (X^*-q(Y_x)) \cup (A \cap q(Y_x))$ is a partition of A. Thus for each $x \in S'$ the sets $A \cap q(X)$ and $A \cap q(Y_x)$ have a common point. The set

$$f^*(A) = f^*(A \cap q(X)) \cup (\bigcup_{x\in S'} f^*(A \cap q(Y_x)))$$

$$= f(A \cap q(X)) \cup (\bigcup_{x\in S'} 1_{Y_x}(A \cap q(Y_x)))$$

$$= \bigcup_{x\in S'} (f(A \cap q(X)) \cup 1_{Y_x}(A \cap q(Y_x))$$

Since the sets $f(A \cap q(X))$ and $1_{Y_x}(A \cap q(Y_x))$ are connected and since their intersection is nonempty, the union $(f(A \cap q(X)) \cup 1_{Y_x}(A \cap q(Y_x))$ is a connected set. Thus $f^*(A)$ being the union of a collection of pairwise intersecting connected sets is a connected set. The proof of the proposition is complete.

Proposition 10.3. If X and Y are T_1-spaces, then f^* is a connectivity function if and only if f is a connectivity function.

Proof. Let $A \subset X^*$ be any connected set. Essentially the following cases arise.

Case I. Either $A \subset q(X)$ or $A \subset q(Y_x)$ for some $x \in S$. Since $f^*/q(X)$ is f and since $f^*/q(Y_x)$ is topologically equivalent to the identity map 1_Y, in either case $G(f^*/A)$ is connected.

Case II. Neither $A \not\subset q(X)$ nor $A \not\subset q(Y_x)$ for each $x \in S$. Let S' denote the set of all points x such that $A \cap q(Y_x) \neq \phi$. We claim that the sets $A \cap q(X)$ and $A \cap q(Y_x)$ are connected. For otherwise any partition of $A \cap q(X)$ or $A \cap q(Y_x)$ will induce a partition of A. Since X and Y are T_1-spaces, the sets $X^*-q(Y_x)$ and $q(Y_x)-\{q(x)\}$

are open in X^*. For each $x \in S'$, $q(x) \in A$. Thus $A \cap q(X)$ and $A \cap q(Y_x)$ have a common point for each $x \in S'$. The set

$$G(f^*/A) = G^*(f^*/(A \cap q(X))) \cup (\bigcup_{x \in S'} G(f^*/(A \cap q(Y_x)))$$

$$= \bigcup_{x \in S'} (G^*(f^*/(A \cap q(X))) \cup G^*(f^*/(A \cap q(Y_x)))$$

Thus $G(f^*/A)$ being the union of a collection of pairwise intersecting connected sets is a connected set.

Proposition 10.4. Let f be a closed (respectively, compact) function and let the set S of singular points be finite. Then f^* is a closed (respectively, compact) function. Conversely, if the function f^* is closed (respectively, compact), then the function f is closed (respectively, compact).

Proof. Suppose f is closed and let $S = \{x_1, \ldots, x_n\}$. Let F be a closed set in X^*. Then

$$f^*(F) = f_1(q^{-1}(F)) = f_1(q^{-1}(F) \cap X) \cup (\bigcup_{i=1}^{n} f_1(q^{-1}(F) \cap Y_{x_i}))$$

$$= f(q^{-1}(F) \cap X) \cup (\bigcup_{i=1}^{n} f_1(q^{-1}(F) \cap Y_{x_i}))$$ where f_1 and q have the same meaning as in the proof of Theorem 2.2. Since the function f is closed, the set $f^*(F)$ being the union of finitely many closed sets is closed.

Suppose f is a compact function and let K be a compact set in Y. Then $f^{*-1}(K) = q(f_1^{-1}(K)) = q(f^{-1}(K) \oplus (\bigoplus_{i=1}^{n} K_{x_i}))$ where $K_{x_i} = K \subset Y_{x_i}$. Since $f^{-1}(K)$ is a compact set and since q is a quotient map, the set $f^{*-1}(K)$ is compact.

CHAPTER III

SOME APPLICATIONS OF OPEN EXTENSIONS

1. In this chapter we point out some applications of the results and techniques of the preceding chapter. The main purpose of this chapter is to show how the results and the techniques of the last chapter can be utilized to obtain analogues and improvements of various results in the folklore of literature on open mappings. As an illustration of this fact we obtain improvements or modifications of results of Arhangelskiǐ, Čoban and Proizvolov on finite-to-one open mappings. Similarly we obtain improvements or analogues of recent theorems of Hodel, Keesling and Nagami on dimension.

This chapter is divided into five sections. In section two finite-to-one open mappings are discussed and it is shown that the hypothesis of "openness" in recent theorems of Arhangelskiǐ, Čoban and Proizvolov can be weakened to some extent. In section three improvements or modifications of recent theorems on open mappings and dimension by Hodel, Keesling and Nagami are obtained. Section four is devoted to the class of mappings which preserve dimension on closed subsets. In section five quasi-open maps are talked of and it is shown that every map can be extended to a quasi-open map in a much easier way than it can be extended to an open map.

2. Finite-to-one mappings

In this section we shall obtain analogues and improvements of some well known theorems on open finite-to-one mappings. Finite-to-one mappings are important because many topological properties are preserved both directly and inversely under open continuous finite-to-one mappings. We shall show that

if a topological property is inversely preserved under open finite-to-one mappings, then it is also preserved under a class of mappings which includes the class of open finite-to-one mappings.

In 1966, Proizvolov [53] showed that if a domain is locally compact, then weight is inversely preserved under open finite-to-one maps. In the same paper he also showed that the hypothesis of local compactness is unessential and that the result also holds if the domain is complete in the sense of Čech (or m-compact or metrizable) (see Theorem I.4.1). Later that year Arhangelskiǐ (see, [3], [4]) showed that weight and metrizability are always inversely preserved under clopen finite-to-one maps. In 1967, Čoban [14] proved that hereditary paracompactness, (metacompactness, Lindelöf property) are inversely preserved under open finite-to-one maps. (Some separation axioms are required for all these results, see Theorems I.4.2, I.4.3, I.4.5 and I.4.6). We obtain the following improvement of these results using the methods of preceding chapter.

Theorem 2.1. *Let f be a continuous finite-to-one mapping of a Hausdorff space X onto a Hausdorff space Y such that f is open except at finitely many points. The following statements are true.*

(a) *If the spaces X and Y are complete in the sense of Čech or m-compact or metrizable, then weight $X \leq$ weight Y.*

(b) *If f is also a closed function, then weight $X \leq$ weight Y.*

(c) *If X is a Tychonoff space, Y a metric space and if f is a closed function, then X is metrizable.*

(d) *If ~~X~~ Y is hereditarily metacompact (respectively, hereditarily Lindelöf), then X is hereditarily metacompact (respectively, hereditarily Lindelöf).*

(e) *If X is regular and if Y is hereditarily paracompact, then X is also hereditarily paracompact.*

Proof. Let the function f^* from X^* onto Y be an open extension of f obtained by the same technique as employed in the proof of Theorem II.2.2. Since f is continuous finite-to-one and since the set S of singular points of X is finite, the functinn f^* is open, continuous and finite-to-one. Sin the set S is finite, the function f^* is closed, whenever f is a closed function (see Proposition II.10.4).

(a) Since the spaces X and Y are complete in the sense of Čech respectively m-compact, metrizable) and since the set S is finite, by Proposition II.8.4 the space X^* is complete in the sense of Čech (respectively, m-compact,metrizable). By Proposition I.4.1 weight X^* weight Y. Since the weight of a space is no larger than the weight c a whole space, weight $X \leq$ weight Y.

(b) Since X and Y are Hausdorff spaces, by Proposition II.4.4 the space X^* is Hausdorff. By Theorem I.4.2 weight X^* = weight Y. Consequer weight $X \leq$ weight Y.

(c) Since X is a Tychonoff space and since Y is metrizable, the space is a Tychonoff space (Proposition II.4.7). By Theorem I.4.3 the space is metrizable. Thus X being a subspace of a metrizable space is metrizable.

(d) Since X and Y are Hausdorff spaces, the space X^* is a Hausdorff space (Proposition II.4.4). By Theorem I.4.5 the space X^* is hereditarily metacompact (hereditarily Lindelöf), whenever Y is so. Thus X is hereditarily metacompact (hereditarily Lindelöf).

(e) Since X is a T_3-space and since Y is Hausdorff hereditarily paracompact space, the space X^* is T_3 (Proposition II.4.6). The result follows by Theorem I.4.6.

Since a locally compact Hausdorff space is complete in the sense of Cech we obtain the following corollary of the Theorem 2.1(a) which is an improvement of a result of Proizovlov (see, Theorem I.4.1).

Corollary 2.2. *Let f be a continuous finite-to-one mapping of a locally compact Hausdorff space X onto a locally compact Hausdorff space Y. If f is open except at finitely many points, then weight $X \leq$ weight Y.*

The following characterization of paracompact p-spaces is due to Arhangelskiǐ (see [2]). It will be useful in proving the next forthcoming theorem. We state it as a lemma.

Lemma (Arhangelskiǐ) 2.3. *A Tychonoff space X is a paracompact p-space if and only if there exists a metric space X and a perfect mapping h of X onto M.*

Using the above characterization of paracompact p-spaces we obtain the following theorem which is a partial improvement of a result of Čoban (Theorem I.4.4).

Theorem 2.4. *Let f be a continuous finite-to-one mapping of a paracompact p-space X onto a metrizable space Y. If f is open except at finitely many points, then X is metrizable.*

Proof. Let the map f^* from X^* onto Y be an open extension of f obtained by the same method as in the proof of Theorem II.2.2. Then f^* is open, continuous and finite-to-one. Since X is a paracompact p-space, there is a metric space M and a perfect map h of X onto M. First we show that the space X^* is a paracompact p-space. Let $M_1 = M \oplus (\bigoplus_{i=1}^{n} Y_{x_i})$ where

$S_f(X) = \{x_1,\ldots,x_n\}$. Let H be the map of W onto W_1 whose restiction to X is the perfect map h and whose restriction to each Y_{x_i} is the identity map $1_{Y_{x_i}}$. Then the map H is a perfect map. Let M^* be the quotient space of W_1 which is obtained by identifying the image $h(x_i)$ of a singular point $x_i \in S_f(X)$ with $f(x_i) \in Y_{x_i}$ and let q^* be the corresponding quotient map of W_1 onto M^*. Then it is easily verified that q^* is a perfect map. Since the space W_1 is metrizable, the space M^* is metrizable (Lemma II.8.3). Let G be a mapping of X^* onto M^* defined by $G(p) = (q^* \circ H)(q^{-1}(p))$ for each $p \in X^*$. Then G is a unique map satisfying $G \circ q = q^* \circ H$. Let F be a closed set in X^*. Then $q^{-1}(F)$ is closed in W and $G(F) = q^* \circ H(q^{-1}(F))$. Therefore, $G(F)$ is a closed set in M^*. Consequently G is a closed map. Since $G^{-1}(m)$ is at most a finite set for each $m \in M^*$, G is a perfect map. Hence the space X^* a paracompact p-space. By theorem I.4.4 the space X^* is metrizable. Since X is a subspace of X^*, it is metrizable. The proof of the theorem is complete.

We do not know whether the hypothesis of paracompactness in theorem 2.4 is unessential.

3. Functions and dimension

In 1963, Hodel (see [31]) proved that for metric spaces dimension cannot be lowered by open maps f such that $f^{-1}(y)$ is discrete. Keesling [36], in 1968, generalized that result by showing that in metric spaces dimension of any subset K of X cannot be lowered by open map f such that $f^{-1}(y)$ is discrete (Theorem I.6.2). Hodel's result is also true for not necessarily continuous functions if Y is taken to be locally compact and separable (Theorem I.6.1). Hodel obtaining a generalization of a result of

Alexandroff showed that if domain is locally compact then dimension is invariant both directly and inversely under open countable-to-one mappings (Theorem I.6.4). We obtain the following improvements or modification of these results.

Theorem 3.1. Let f be a function from a metric space X onto a metric space Y such that singular points of X do not accumulate. The following statements are true.

(a) *If f is continuous and if $f^{-1}(y)$ is discrete for each $y \in Y$, then for any subset K of X $\dim K \leq \dim f(K)$.*

(b) *If Y is locally compact separable and if $f^{-1}(y)$ is discrete for each $y \in Y$, then $\dim X \leq \dim Y$.*

(c) *Let X and Y be locally compact and let f be countable-to-one. If f is continuous and if the set S of singular points is countable, then $\dim X \leq \dim Y$.*

Proof. Let the function f^* from X^* onto Y be an open extension of f obtained by the same technique as employed in the proof of Theorem II.2.2. Since X and Y are metric spaces and since the singular points of X do not accumulate, by Proposition II.8.4 the space X^* is metrizable.

(a) Since f is continuous, f^* is continuous (Theorem II.2.2). Since $f^{-1}(y)$ is discrete for each $y \in Y$, $f^{*-1}(y)$ is discrete for each $y \in Y$. By Theorem I.6.2 $\dim K \leq \dim f^*(K)$ for any subset K of X^*. Hence $\dim K \leq \dim f(K)$ for any subset K of X.

(b) By Theorem I.6.1 $\dim X^* \leq \dim Y$. Since X is a closed subspace of X, $\dim X \leq \dim Y$.

(c) Since X and Y locally compact and since singular points of X do not accumulate, by Proposition II.8.4 X^* is locally compact. Since f is continuous, countable-to-one and since the set S of singular

points of X is countable, f^* is continuous and countable-to-one. By Theorem I.6.4 $\dim X^* = \dim Y$. Since X is a closed subspace of X^*, $\dim X \leq \dim Y$.

Keesling [38] showed that for Tychonoff spaces dimension is preserved both directly and inversely under clopen finite-to-one mappings (see Theorem I.6.6). Nagami [49] proved that in the class of paracompact spaces dimension is preserved both directly and inversely under open finite-to-one mappings. Moreover, he showed that same result also holds for large inductive dimension in the class of hereditarily paracompact spaces (see Theorem I.6.6). We obtain the following one sided improvements of these results.

Theorem 3.2. *Let X and Y be Tychonoff spaces and let f be a continuous function from X onto Y such that f is open except at finitely many points. Then the following holds true.*

(a) *If f is also a closed function, then $\dim X \leq \dim Y$.*

(b) *If the spaces X and Y are paracompact and if the function f is finite-to-one, then $\dim X \leq \dim Y$.*

(c) *If the spaces X and Y are hereditarily paracompact and if the mapping f is finite-to-one, then $\operatorname{Ind} X \leq \operatorname{Ind} Y$.*

Proof. Let the function f^* from X^* onto Y be an open extension of f obtained by the same method as in the proof of Theorem II.2.2. Since X and Y are Tychonoff space, the space X^* is a Tychonoff space (Proposition II.4.7).

(a) Since f is a closed function and since the set S of singular points is finite, the function f^* is closed (Proposition II.10.4). By Theorem I.6.6(a) $\dim X^* = \dim Y$. Let $g \in C(X)$ be any function. Extend g to a continuous realvalued function G by taking G to be constantly $g(x)$ on Y_x for each $x \in S$. The function G in its turn induces a

continuous realvalued function on X^*. Thus X is C-embedded in X^*. By Theorem I.5.5 $\dim X \leq \dim X^*$. Therefore, $\dim X \leq \dim Y$.

(b) Since X and Y are paracompact space, by Proposition II.5.3 the space X^* is paracompact. Since f is finite-to-one and since the set S is finite, the function f^* is finite-to-one. By Theorem I.6.6 $\dim X^* = \dim Y$. Since X^* is a paracompact Hausdorff space and since X is a closed subspace of X^*, $\dim X \leq \dim X^*$ (see Theorem I.5.5).

Proof of (c) is similar to the proof of (b) and in the proof of (c) we make use of Theorem I.6.6 (d) and the fact that $\operatorname{Ind} X \leq \operatorname{Ind} X^*$.

4. Functions preserving dimension on closed subsets

In 1968, Keesling (see [36]) pointed out a class of functions which preserve dimension on closed subsets. In this section we show that a wider class of functions preserve dimension on closed subsets.

The following theorem represents improvements or analogues of results due to Keesling (see Theorem I.6.5).

Theorem 4.1. *Let f be a continuous function from a metric space X onto a metric space Y such that the singular points of X do not accumulate. The following statements are true.*

(a) *Let the function f be σ-closed. If $f^{-1}(y)$ is discrete for each $y \in Y$ and if the set S of singular points of X is countable, then f is dimension preserving on closed subsets.*

(b) *Let the spaces X and Y be separable. If $f^{-1}(y)$ is discrete for each $y \in Y$ and if the set S is countable, then f is dimension preserving on closed subsets.*

(c) If f is finite-to-one and if the set S of singular points of X is finite, then f is dimension preserving on closed subsets.

Proof. Let the function f^* from X^* onto Y be an open extension of f obtained by the same technique as employed in the proof of Theorem II.2.2. Since X and Y are metric space and since the singular points of X do not accumulate, the space X^* is metrizable (Theorem II.8.4).

(a) Since $f^{-1}(y)$ is discrete for each $y \varepsilon Y$, $f^{*-1}(y)$ is discrete for each $y \varepsilon Y$. Since the function f is σ-closed and since the set S is countable, the function f^* is σ-closed. By Theorem I.6.5(a) the function f^* preserves dimension on closed subsets. Since X is a closed subspace of X^*, the function f also preserves dimension on closed subsets.

(b) Since the spaces X and Y are separable and since the set S of singular point of X is countable, the space X^* is separable (Proposition II.7.3). Since $f^{-1}(y)$ is discrete for each $y \varepsilon Y$, $f^{*-1}(y)$ is discrete for each $y \varepsilon Y$. By Theorem I.6.5(b) the function f^* preserves dimension on closed subsets. Since X is a closed subspace of X^*, the function f preserves dimension on closed subsets.

(c) Since the function f is finite-to-one and since the set S is finite, the function f^* is finite-to-one. By Theorem I.6.5(c) the function f^* preserves dimension on closed subsets. Thus the function f preserves dimension on closed subsets.

Keesling in [38] showed that for normal spaces clopen maps preserve dimension on closed subsets (see, Theorem I.6.6(b)). We obtain the following improvement of this result.

Theorem 4.2. Let f be a continuous closed function from a T_4-space X onto a T_4-space Y. If f is open except at finitely many points, then f is

dimension preserving on closed subsets.

In the proof of Theorem 4.2 we make use of Proposition II.5.3 and Theorem I.6.6(b).

5. Quasi-open extensions

The notion of quasi-openness was first introduced by Whyburn (see[56]) in a study of open and related mappings on locally compact metric spaces and their applications to analytic functions. Since any map f from a topological space X into a topological space Y can be extended to an open map f^* from a topological space X^* onto Y (Theorem II.2.2) and since on open map is quasi-open, any map f can be extended to a quasi-open map. Nevertheless it may be of some interest (or useful) to know whether a map can be extended to a quasi-open map more easily (or conveniently or economically in some sense) than it can be extended to an open map. In this section we show that a slight modification of the technique employed in chapter II for extending a map to an open map yields a quasi-open extension. In general the open extension and the quasi-open extension are distinct, and we give a necessary and sufficient condition for the two extensions to be the same.

Definition 5.1. Let f be a function from a topological space X into a topological space Y. The function f is said to be quasi-open provided for any open set U in X containing a compact component of $f^{-1}(y)$, y is an interior point of $f(U)$.

Clearly every open function is quasi-open but not conversely. The function f in Example II.6.1 is quasi-open but not open. However, it is obvious that if f is both quasi-open and light (a map f from X to Y is light if $f^{-1}(y)$ is totally disconnected for every $y \in Y$), it is necessarily open.

Definition 5.2. A component K of $f^{-1}(y)$ is called a singular component of f if there is a neighbourhood U of K such that $f(U)$ is not a neighbourhood of y. A point $x \in X$ is called a strong singular point of X if x lies in a singular component of f.

Thus clearly every strong singular point is a singular point but not conversely. Let S^{**} denote the set of strong singular points of X. The function f is quasi-open if and only if the set S^{**} of strong singular points of X is empty. Now if the set S^{**} is not empty, then we can pick exactly one element from each singular component of f. Let S^* denote a choice set. Then the set S^* is unique up to cardinality and is a subset of S^{**}.

Example 5.3. We show that the sets S, S^* and S^{**} are distinct.

Let $X = \{(x,0): -1 \leq x \leq 1\} \cup \{(0,y): -1 \leq y \leq 1\} \cup \{(0,y): 2 \leq y \leq 3\}$, $Y = [-1,1]$, where X and Y have the relative topology from the plane and f is the restriction of the usual projection. Then the set S^* is a singleton, the set $S^{**} = \{(0,y) : 2 \leq x \leq 3\}$ and the set S consists of all the points on the y-axis except the origin.

The construction of a quasi-open extension 5.4. Let $W^* = X \oplus (\bigoplus_{x \in S^*} Y_x)$ where the second disjoint topological sum is taken over all $x \in S^*$. By identifying each point $x \in S^* \subseteq X$ (with X thought of as a subset of W^*) by its image $f(x)$ (as a point of $Y_x \subseteq W$) we arrive at a quotient space X^{**} of W^*. Let q^* denote the corresponding quotient map. Then the inclusion map i_X from X into W composes with the quotient map q^* to give a homeomorphic embedding of X into X^{**}. Hence we may think of X^{**} as a superspace of X.

Let f_1 from W onto Y be the function whose restriction to X is f and whose restriction to each Y_x is the identity map of Y_x onto Y.

then there is a unique function f^{**} from X^{**} onto Y satisfying $f^{**} \circ q = f_1$ which is a quasi-open extension of f.

Since in general $S \neq S^*$, the two extensions viz., the open extension (discussed in chapter II) and quasi-open extension are distinct. If a singular component of f is nondegenerate, then S^* is a proper subset of S^{**} and hence a proper subset of S. Thus $S^* = S^{**}$ for all choices of S^* if and only if each singular component of f is a singleton. The set $S = S^{**}$ if and only if each singular point lies in a singular component of f. Therefore, $S = S^*$ for all choices of S^* if and only if each singular point of X is a singular component of f. Hence the two constructions coincide if and only if each singular point of X is a singular component of f. In particular, the two constructions coincide if f is a light mapping.

CHAPTER IV

OTHER OPEN EXTENSION CONSTRUCTIONS

1. In this chapter we give two more methods of extending a continuous function to an open continuous function. It follows from the first method of construction that every continuous function from a topological space into another is the restriction of a projection. The same method of construction also implies that any continuous function from a topological group (respectively, topological vector space) into another is the restriction of an open continuous homomorphism (respectively, linear mapping). Although this method of construction seems only of an intrinsic interest and has no applications as such. In the second method of construction, we give a method of unifying the domain and range of a continuous function so as to yield a meaningful open continuous extension of the function. We also relate the topological properties of domain and range with the domain of extension. A slight modification of the second method of construction is useful in obtaining partial improvements or modifications of recent theorems of Čoban and Proizvolov.

2. Projection and open extensions.

In this section we present a new method of extending a continuous function to an open continuous function.

Construction 2.1. Let f be a continuous function from a topological space X into a topological space Y. Let 1_X denote the identity mapping of X into itself. Since the family $\{1_X, f\}$ of continuous functions distinguishes points and, distinguishes points and closed sets, by the

embedding lemma (Theorem I.3.1) the evaluation map e from X into Y (defined by $e(x) = (x,f(x))$) is an embedding. Hence X is homeomorphic to $e(X)$, the graph of f and the co-ordinate projection π_Y from $X \times Y$ onto Y restricted to $e(X)$ is topologically equivalent to f. Thus each continuous function is the restriction of a projection map and hence also of an open continuous function. If X is compact, then f is the restriction of a continuous clopen function. This also holds if X is countably compact and Y is a subspace of a sequential space (see [21]). Moreover, if X and Y are topological groups (respectively, topological vector spaces), then the projection π_Y is a homomorphism (respectively, linear mapping). Thus each continuous function from a topological group (respectively, topological vector spaces) into another is the restriction of a continuous homomorphism (respectively, linear mapping).

Combining the results of the preceding paragraph together with the results that of chapter II, we obtain the following result.

Theorem 2.2. Let f be a continuous function from a topological space X into a topological space Y. Then there is a superspace $\tilde{X}$ of X and an open continuous function $\tilde{f}$ from $\tilde{X}$ onto Y whose restriction to X is the function f. Moreover, the space $\tilde{X}$ may be chosen to satisfy any one of the following properties of X and Y.

1. Any finitely productive property. For example,
   - (a) the separation axioms T_0, T_1, T_2, regularity and complete regularity,
   - (b) compactness, sequential compactness, realcompactness and local compactness.

(c) both axioms of countability, weight, local weight, separability, density and metrizability,

(d) connectedness, pathwise connectednes, total disconnectedness and contractibility.

2. Any co-reflexive property. For example,

(a) being a sequential space, k-space, c-space, chain net space or a P-space,

(b) being locally (pathwise) connected.

3. The following normality concepts with T_1 axiom.

(a) normality, hereditary normality and perfect normality,

(b) collectionwise normality, full normality (= paracompactness), m-paracompactness and normality, where m is an infinite cardinal number.

Clearly, any finitely productive property of the domain and range of the map is preserved in the domain of the projection. Properties 2. and 3. however, are not amenable to the method of construction of this section. If H is the real line with the half-open interval topology, one can define a map from H into H which essentially shrinks a closed interval to a point. Here the method of construction of this section doesn't preserve, normality and paracompactness. To show that the method of construction of this section is not amenable to all the properties 3.,we first prove the following proposition.

Proposition 2.3. Let X be a countable Hausdorff space. Then the following statements are equivalent.

(a) The space X is a k-space.

(b) The space X is a sequential space.

(c) The space X is a chain net space.

Proof. The equivalence (a) $\Longleftrightarrow$ (b) was established by Arhangelskiĭ and Franklin (see [5], Proposition 1.5). The implication (b) $\Longleftrightarrow$ (c) is obvious. To show (c) $\Longleftrightarrow$ (b), suppose X is a chain net space. Let A be a sequentially closed set in X and let Φ be a well ordered net from D into A which converges to a point $x \in A$. If D is countable, then it contains a cofinal sequence. Consequently $x \in A$. Let D be uncountable. Since X is countable, the range of Φ is countable. Let $\Phi(D) = \{x_1, x_2, \ldots\} \subseteq A$. For each $i \in \mathbb{N}$ consider $\Phi^{-1}(x_i)$. If $\Phi^{-1}(x_i)$ is cofinal in D for some i, then Φ is frequently constant. Since X is Hausdorff, $x \in A$. If $\Phi^{-1}(x_i)$ is not cofinal for any $i \in \mathbb{N}$, then for each $i \in \mathbb{N}$ there is an $\alpha_i \in D$ which succeeds $\Phi^{-1}(x_i)$. Since A is sequentially closed, if the sequence $\{\alpha_i\}_{i \in \mathbb{N}}$ is cofinal in D, then $x \in A$. If $\{\alpha_i\}_{i \in \mathbb{N}}$ is not cofinal in D, there is an $\alpha \in D$ such that α succeeds $\{\alpha_i : i \in \mathbb{N}\}$. This is a contradiction. Therefore A is a closed set in X. Thus X is a sequential space. The proof is complete.

Let Q denote the space of rationals with the relative topology from the real line. Let f be the quotient map from Q onto Q/Z (in the topological sense; the integers are shrunk to a point). Then the product space $Q \times Q/Z$ is not a sequential space (see [22], Example 1.11). By Proposition 2.3 the space $Q \times Q/Z$ is neither a k-space nor a chain net space. Thus the domain of projection need not preserve the property of being a k-space, sequential space or a chain net space.

3. A Unified Open Extension

Let f be a function, not necessarily continuous, from a topological space X into a topological space Y. Let S and T denote the sets of singular points of X and Y respectively. Without any loss of generality, we may assume that the spaces X and Y are disjoint. Let W denote the set theoretic union of X and Y. Define a subset O of W to be open if it satisfies the following conditions.

(i) The sets $O \cap X$ and $O \cap Y$ are open in X and Y respectively.

(ii) The set $O \cap S = \phi$ or else for each $x \in O \cap S$, the set $O \cap Y$ contains a neighbourhood of $f(x)$ in Y.

Let $\mathcal{T}$ denote the collection of all open sets (i.e. all sets satisfying (i) and (ii)) so defined. Then we have the following proposition.

<u>Proposition</u> 3.1. <u>The collection $\mathcal{T}$ is a topology for W.</u>

<u>Proof.</u> It follows from the definition that the empty set and the space W are open. Let $\{O_\alpha\}_{\alpha\in I}$ be any collection of open sets and let $Q = \bigcup_{\alpha\in I} O_\alpha$. Then the set $Q \cap X = \bigcup_\alpha (O_\alpha \cap X)$ and the set $Q \cap Y = \bigcup_\alpha (O_\alpha \cap Y)$ are open in X and Y respectively. In order to show that Q satisfies condition (ii), let $x \in Q \cap S$. Then $x \in O_\alpha \cap S$ for some $\alpha \in I$. Since O_α is an open set, the set $O_\alpha \cap Y$ contains a neighbourhood of $f(x)$. This in turn implies that $Q \cap Y$ contains a neighbourhood of $f(x)$. Thus the union of any collection of open sets is open.

Let G_1 and G_2 be any two open sets and let $G = G_1 \cap G_2$. Then the sets $G \cap X = (G_1 \cap X) \cap (G_2 \cap X)$ and $G \cap Y = (G_1 \cap Y) \cap (G_2 \cap Y)$ are open in X and Y respectively. In order to show that G satisfies the condition (ii), let $x \in G \cap S$. Then $x \in G_1 \cap S$ and $x \in G_2 \cap S$. Since the sets

G_1 and G_2 are open, there are open neighbourhoods N_1 and N_2 (in Y) of $f(x)$ such that $N_1 \subset G_1 \cap Y$ and $N_2 \subset G_2 \cap Y$. Then, the intersection $N_1 \cap N_2$ is a neighbourhood of $f(x)$ contained in $G \cap Y$. This completes the proof of the proposition.

<u>Proposition</u> 3.2. <u>The set X is closed in W and the set Y is open in W.</u>

<u>Proof.</u> Since Y satisfies conditions (i) and (ii), Y is an open set in W. The set X being the complement of Y is open in W.

<u>Proposition</u> 3.3. <u>The retraction map r from W onto Y, defined by $r(x) = f(x)$ for $x \varepsilon X$ and $r(x) = x$ for $x \varepsilon Y$ is an open function. If the function f is continuous, so also is r.</u>

<u>Proof.</u> Let V be an open set in W. Then $r(V) = r(V \cap X) \cup r(V \cap Y) = f(V \cap X) \cup (V \cap Y)$. The set $V \cap Y$ is open in Y. If $f(V \cap X)$ is open in Y, the set $r(V)$ being the union of two open sets is an open set. On the other hand let $f(V \cap X)$ be not open and suppose it is not a neighbourhood of its point y. Then $y = f(x)$ for some $x \varepsilon V \cap X$ and x is a singular point of X. Since V is open in W, the set $V \cap Y$ contains a neighbourhood of $f(x)$. Thus the set $r(V)$ being a neighbourhood of each of its points is an open set.

To prove the remaining part of the proposition, suppose f is continuous and let U be an open set in Y. Then $r^{-1}(U) = f^{-1}(U) \cup U$ and $r^{-1}(U)$ satisfies the conditions (i) and (ii). Therefore, $r^{-1}(U)$ is open in W. This completes the proof of the proposition.

<u>Proposition</u> 3.4. <u>If the function f is closed (respectively, compact), then the retraction map r as defined in the Proposition 3.3 is a closed (respectively, compact) function.</u>

Proof. Let F be a closed set in W. Then the sets $F \cap X$ and $F \cap Y$ are closed in X and Y respectively. The set $r(F) = r(F \cap X) \cup r(F \cap Y) = f(F \cap X) \cup F \cap Y$. Since F is a closed function, the set $r(F)$ being the union of two closed sets is a closed set.

Let K be a compact set in Y. Then $r^{-1}(K) = f^{-1}(K) \cup K$. Since f is a compact function, $f^{-1}(K)$ is a compact set and $r^{-1}(K)$ being the union of two compact sets is a compact set. This completes the proof of the proposition.

The following theorem summarizes the results of the preceding paragraphs.

Theorem 3.5. Let f be a function, not necessarily continuous, from a topological space X into a topological space. Let W denote the disjoint set theoretic union of X and Y. Then there is a topology on W such that the following are satisfied.

(a) The space X is a closed subspace of W and the space Y is an open subspace of W.

(b) The retraction map r from W onto Y defined by $r(z) = f(z)$ for $z \in X$ and $r(z) = z$ for $z \in Y$ is an open function.

(c) The retraction map r is a continuous function (respectively, closed function, compact function) if and only if f is a continuous function (respectively, closed function, compact function).

Let W be topologized with the topology $\mathcal{T}$ defined by the conditions (i) and (ii). Then the conditions (a) - (c) of theorem 3.5 are satisfied. There may exist more than one topologies on W which satisfy the conditions of Theorem 3.5. For example, let the condition

(ii) be replaced by (ii)' : the set $Q \cap S = \phi$ or else for each $x \in Q \cap S$, the set $Q \cap Y$ contains a deleted neighbourhood of $f(x)$ in Y. Then the conditions (i) and (ii)' defines a topology T' on W such that $T \subset T'$ and the conditions (a) - (c) of theorem 3.5 are satisfied. If the function f is same as in example II.6.1, then $T = T'$. Thus in general the topologies T and T' do not coincide

We do not know whether T is the smallest topology on W satisfying conditions (a) - (c) of theorem 3.5.

Throughout this section f will stand for a function from a topological space X into a topological space Y and W will denote the space obtained from the disjoint union of the sets X, Y and topologized with the topology T as defined above. If the function f is open, then the space W is the disjoint topological sum $X \oplus Y$. In the discussion that follows we assume that f is not open so that the space W is different from the disjoint topological sum $X \oplus Y$.

Proposition 3.6. The space W is T_0 if and only if X and Y are T_0.

Proof. Since the spaces X and Y are subspaces of W and since the property of being a T_0-space is hereditary, the necessity follows. To prove sufficiency suppose x_1 and x_2 are any two distinct points of W. Since Y is T_0, if x_1 and x_2 both lie in Y, then there is a neighbourhood N(in Y) of one of x_1 or x_2 which does not contain the other. Since Y is open in W, N is a neighbourhood (in W) of one of x_1 or x_2 which does not contain the other. If $x_1 \in X$ and $x_2 \in Y$, then Y is a neighbourhood of x_2 which does not contain x_1. If both of the points x_1 and x_2 are in X, then since X is T_0, there is a neighbourhood N(in X) of one of x_1 and x_2 which does not contain the other. Suppose N is a

neighbourhood of x_1. Then $N \cap Y$ is a neighbourhood in W of x_1 which does not contain x_2. This completes the proof of the proposition.

Proposition 3.7. The space W is never T_1 and hence never T_2. Furthermore, the space W is never regular.

Proof. Let $x \in S$ be any singular point of X. Then no neighbourhood (in W) of x is free from $y = f(x)$. Thus W is not T_1 and hence not T_2. Further, if $y \in T$ is any singular point of Y, then the closed set X and the point y cannot be separated by disjoint open sets. Therefore, the space W is not regular.

Proposition 3.8. Let P be any one of the following properties.

(1) compactness (2) countable compactness (3) sequential compactness (4) pseudocompactness (5) (hereditary) m-Lindelöf property (6) hereditary m-separability.

Then the space W has property P, if X and Y have property P.

Proof. Suppose X and Y have property P. Since the property P is preserved under finite disjoint topological sums, the disjoint topological sum $X \oplus Y$ has property P. Since the space W is the continuous image of the space $X \oplus Y$ and since the property P is preserved under continuous surjections, the proposition follows.

Proposition 3.9. The space W is connected if the spaces X and Y are connected.

Proof. Suppose X and Y are connected. Let $W = X_1 \cup X_2$ be a partition of W. Then $X = (X_1 \cap X) \cup (X_2 \cap X)$ and $Y = (X_1 \cap Y) \cup (X_2 \cap Y)$. Since X and Y are connected, either $X_1 \cap X = \phi$ or $X_2 \cap X = \phi$ and either $X_1 \cap Y = \phi$ or $X_2 \cap Y = \phi$. Suppose $X_1 \cap X = \phi$. Then $X_2 \cap X = X$. Since X_2 is open in W, for each singular point x of X there is a neighbourhood

N_x of $f(x)$ such that $N_x \subset X_2$. Consequently, $X_2 \cap Y \neq \phi$. Therefore, $X_1 \cap Y = \phi$. Hence $X_1 = \phi$. This completes the proof of the proposition.

Proposition 3.10. *If X and Y are locally connected, then the space W is locally connected.*

Proof. Let $x \in W$ be any point and let V be a neighbourhood (in W) of x. The following two cases arise.

Case I. The point $x \in Y$. Since Y is locally connected and since Y is open in W, there is a connected neighbourhood U of x in W such that $U \subset V$.

Case II. The point $x \in X$. Since $V \cap X$ is a neighbourhood of x in X and since X is locally connected, there is a connected neighbourhood U of x (in X) such that $U \subset V \cap X$. If U contains no singular point of X, then U is open in W and we are done. Otherwise, for each singular point $x \in U \cap S$, there is a neighbourhood N_x of $f(x)$ (in Y) which is contained in V. Since Y is locally connected, we can choose N_x to be connected. Let $N = U \cup (\bigcup_x N_x)$, where the second union is taken over all $x \in U \cap S$. Then N is a connected neighbourhood of x in W and $N \subset V$. The proof of the proposition is complete.

The Proposition 3.7 tells us that the space W is never a T_1-space and hence never a T_2-space. Nevertheless, in some cases it is possible to reduce the new domain to a T_1-space or even to a T_2-space by deleting some of the points of W. The next proposition tells us under what conditions on the function f and on the sets of singular points of X and Y respectively, W can be reduced to a T_1-space or a T_2-space.

In the discussion that follows, f will denote a function from a topological space X into a topological space Y as before. The letters

S and T will stand for the sets of singular points of X and Y respectively.

Proposition 3.11. *If singular points of X and Y do not accumulate, f/S is one-to-one and if $\bar{X} = W-T$, then (1) The space $\bar{X}$ is T_1 if the spaces X and Y are T_1, (2) The space $\bar{X}$ is T_2 if the spaces X and Y are T_2, (3) The function $\bar{f} = f/\bar{X}$ is open, and (4) The function $\bar{f}$ is continuous if f is.*

Proof. (1) Suppose that the spaces X and Y are T_1 and let x_1, x_2 be any two distinct points of $\bar{X}$. The following three cases arise.

Case I. Both of the points x_1 and x_2 are in X. Since X is a T_1-space, there are neighbourhoods N_1 and N_2 (in X) of x_1 and x_2 respectively such that $x_1 \notin N_2$ and $x_2 \notin N_1$. Therefore, the sets $(N_1 \cup Y) \cap \bar{X}$ and $(N_2 \cup Y) \cap \bar{X}$ are neighbourhoods in X of x_1 and x_2 respectively such that $x_2 \notin (N_1 \cup Y) \cap \bar{X}$ and $x_1 \notin (N_2 \cup Y) \cap \bar{X}$.

Case II. One of the points x_1 and x_2 lies in X and the other lies in Y. Suppose $x_1 \in X$. Then $x_2 \in \bar{X}-X$. Since $\bar{X} = W-T$, $x_2 \in Y-T$. If x_1 is a nonsingular point of X, then by hypothesis on the set S of singular points of X, there is a neighbourhood N(in X) of x_1 which contains no singular point of X. Thus N and $\bar{X}-X$ are disjoint neighbourhoods in $\bar{X}$ of x_1 and x_2 respectively. If x_1 is a singular point of X, then $f(x_1) \neq x_2$ and by the hypothesis on the set S of singular points of X there exists a neighbourhood N(in X) of x_1 which contains no other singular points of X. Since X is a T_1-space, there exists a neighbourhood N_1(in Y) of $f(x_1)$ such that $x_2 \notin N_1$. Then $(N \cup N_1) \cap \bar{X}$ and $Y-T$ are neighbourhoods in $\bar{X}$ of x_1 and x_2 respectively such that $x_2 \notin (N \cup N_1) \cap \bar{X}$ and $x_1 \notin Y-T$.

Case III. Both of the points x_1 and x_2 are in Y. Since Y is a T_1-space, there are neighbourhoods N_1 and N_2 (in Y) of x_1 and x_2 respectively such that $x_2 \notin N_1$ and $x_1 \notin N_2$. Since Y is open in W, the sets $N_1 \cap \overline{X}$ and $N_2 \cap \overline{X}$ are neighbourhoods in $\overline{X}$ of x_1 and x_2 respectively.

(2) Suppose that the spaces X and Y are T_2. Let x_1 and x_2 be only two distinct point of $\overline{X}$. The following three cases arise.

Case I. Both of the points x_1 and x_2 are in X. Since X is T_2, there are disjoint neighbourhoods N_1 and N_2 (in X) of x_1 and x_2 respectively. By hypothesis on the set S of singular points of X, the neighbourhoods N_1 and N_2 can be so chosen that they contain no other singular point of X. If the points x_1 and x_2 are nonsingular points of X, then N_1 and N_2 are disjoint neighbourhoods in $\overline{X}$ of x_1 and x_2 respectively. If one of the points x_1 and x_2 is a singular point and the other is a nonsingular point of X, suppose x_1 is a singular point, then $(N_1 \cup Y)-T$ and N_2 are disjoint neighbourhoods in $\overline{X}$ of x_1 and x_2 respectively. Since f/S is one-to-one, if both of the points x_1 and x_2 are singular points of X, then $f(x_1) \neq f(x_2)$. Since Y is a T_2-space, there exist disjoint neighbourhoods N_1' and N_2' (in Y) of $f(x_1)$ and $f(x_2)$ respectively. Then $(N_1 \cup N_1') \cap \overline{X}$ and $(N_2 \cup N_2') \cap \overline{Y}$ are disjoint neighbourhoods in $\overline{X}$ of x_1 and x_2 respectively.

Case II. Only one of the points x_1 and x_2 belongs to X. Suppose $x_1 \in X$. Then $x_2 \in \overline{X}-X = Y-T$. By hypothesis on the set S of singular points of X, there exists a neighbourhood N (in X) of x_1 which contains no other singular point of X. If x_1 is a nonsingular point of X,

then N and $Y-T$ are disjoint neighbourhoods in $\overline{X}$ of x_1 and x_2 respectively. If x_1 is a singular point of X, then $f(x_1) \notin \overline{X}$. Hence $f(x_1) \neq x_2$. Since Y is a T_2-space, there are disjoint neighbourhoods N_1 and N_2 in Y of $f(x_1)$ and x_2 respectively. Consequently, $(N \cup N_1) \cap \overline{X}$ and $N_2 \cap \overline{X}$ are disjoint neighbourhoods in $\overline{X}$ of x_1 and x_2 respectively.

Case III. Both of the points x_1 and x_2 are in Y. Since Y is a T_2-space, there exist disjoint neighbourhoods N_1 and N_2 in Y of x_1 and x_2 respectively. Since Y is open in W, $N_1 \cap \overline{X}$ and $N_2 \cap \overline{X}$ are disjoint neighbourhoods in $\overline{X}$ of x_1 and x_2 respectively.

(3) Since the singular points of Y do not accumulate, the set T is closed in Y. Since Y is open in W, the set $Y-T$ is open in W. In order to show that the function $\overline{f}$ is open, let V be an open set in $\overline{X}$. Then there exists an open set U in W such that $V = U \cap (W-T)$. The set $\overline{f}(V) = \overline{f}(V \cap X) \cup \overline{f}(V \cap (Y-T)) = f(V \cap X) \cup (V \cap (Y-T))$. If $f(V \cap X)$ is open in Y, the set $\overline{f}(V)$ being the union of two open sets is an open set. Suppose $f(V \cap X)$ is not a neighbourhood of its point y. Then $y = f(x)$ for some $x \in V \cap X$ and x is a singular point of X. Since U is open in W there is a neighbourhood N in Y of y such that $N \subset U \cap Y$. By the hypothesis on the set T of singular points of Y, N can be so chosen that it contains no singular points of Y other than y. Then $N-\{y\} \subset V \cap (Y-T)$. Therefore, $N \subset f(V \cap X) \cup (V \cap (Y-T)) = \overline{f}(V)$. Since the set $\overline{f}(V)$ contains a neighbourhood of each of its points, it is an open set.

(4) Since f is continuous, the retraction map r is continuous (see Proposition 3.3) and the function $\overline{f}$ being the restriction of r to $\overline{X}$

is continuous. The proof of the proposition is complete.

Proposition 3.12. *If the singular points of X and Y do not accumulate, f/S is one-to-one and $\overline{X} = W-T$ then the space $\overline{X}$ is locally compact Hausdorff if and only if X and Y are locally compact Hausdorff.*

Proof. Since the spaces X and Y are Hausdorff and since the hypothesis of Proposition 3.11 is satisfied, it follows that the space $\overline{X}$ is Hausdorff. Let $x \varepsilon \overline{X}$ be any point. Now the following two cases arise.

Case I. The point $x \varepsilon X$. Since the space X is locally compact, there is a compact neighbourhood N (in X) of x. By hypothesis on the set of S of singular points of X, the neighbourhood N can be so chosen that it contains no other singular point of X. If x is a nonsingular point of X, then N is a compact neighbourhood of x in W. If x is a singular point of X, then $f(x)$ is a singular point of Y. Since Y is locally compact and since the singular points of Y do not accumulate, there is a compact neighbourhood N_x (in Y) of $f(x)$ which contains no other singular points of Y. Then $N \cup N_x$ is a compact neighbourhood of x in W and $(N \cup N_x) \cap \overline{X} = (N \cup N_x)-\{f(x)\}$ is a neighbourhood of x in $\overline{X}$. We shall show that the set $(N \cup N_x)-\{f(x)\}$ is compact. Let $\{x_\alpha\}_{\alpha \varepsilon D}$ be a net in $(N \cup N_x)-\{f(x)\}$. Since $N \cup N_x$ is compact, the net $\{x_\alpha\}_{\alpha \varepsilon D}$ has a cluster point $p \varepsilon N \cup N_x$. If $p \neq f(x)$ then $p \varepsilon (N \cup N_x)-\{f(x)\}$. If $p = f(x)$, then x is also a cluster point of the net $\{x_\alpha\}_{\alpha \varepsilon D}$. In either case the net $\{x_\alpha\}_{\alpha \varepsilon D}$ has a cluster point in $(N \cup N_x)-\{f(x)\}$. So $(N \cup N_x)-\{f(x)\}$ is a compact neighbourhood of x (in $\overline{X}$).

Case II. The point $x \varepsilon \overline{X}-X$. Since Y is locally compact and since the singular points of Y do not accumulate, there is a compact neighbourhood N of x in Y which contains no singular points of Y. Then

$N = N \cap \bar{X}$ is a compact neighbourhood of x in $\bar{X}$. The proof of the proposition is complete.

<u>Proposition</u> 3.13. <u>Let</u> X <u>and</u> Y <u>be</u> T_3<u>-spaces and let</u> f/S <u>be one-to-one. If singular points of</u> X <u>and</u> Y <u>do not accumulate, then</u> $\bar{X}$ <u>is a</u> T_3<u>-space.</u>

<u>Proof.</u> Let F be a closed set in $\bar{X}$ and let $x \notin F$ be any any point of $\bar{X}$. The following two cases arise.

<u>Case</u> I. The point x is a singular point of X. Since F is closed in $\bar{X}$, the set $X \cap F$ is closed in X and the set $F \cap Y$ is closed in Y-T. Now the point $f(x)$ cannot be a limit point (in Y) of $F \cap Y$. For, if $f(x)$ is a limit point (in Y) of $F \cap Y$, then since x is a limit point in W of $f(x)$, the point x is a limit point of F in $\bar{X}$. Therefore, the set $(F \cap Y) \cup (T-\{f(x)\})$ is closed in Y. Since Y is T_3, there exist disjoint open sets N_1 and N_2 in Y containing $f(x)$ and $(F \cap Y) \cup (T-\{f(x)\})$ respectively. Since X is T_3, there exist disjoint open sets N_3 and N_4 in X containing x and $F \cap X$ respectively. Then $(N_1 \cup N_3) \cap \bar{X}$ and $(N_2 \cup N_4) \cap \bar{X}$ are disjoint neighbourhoods in $\bar{X}$ of x and F respectively.

<u>Case</u> II. The point x is not a singular point of X. If $x \in X$, then since X is T_3, there are disjoint neighbourhoods N_1 and N_2 in X of x and $F \cap X$ respectively. (in case $F \cap X = \phi$, let $N_2 = \phi$). By the hypothesis on the set S of singular points of X, N_1 can be chosen so that it contains no singular point of X. Then N_1 and $(N_2 \cup Y) \cap \bar{X}$ are disjoint neighbourhoods in $\bar{X}$ of x and F respectively. If $x \in \bar{X}$-X, then since $F \cap Y$ is closed in Y-T and since T is closed in Y, the set $T \cup (F \cap Y)$ is closed in Y. Since Y is T_3, there are disjoint neighbourhoods N_1 and N_2 in Y of x and $T \cup (F \cap Y)$ respectively. Then $N_1 \cap \bar{X}$ and $(X \cup N_2) \cap \bar{X}$ are

disjoint neighbourhoods in $\bar{X}$ of x and F respectively. The proof of the proposition is complete.

In III.2 we showed that hypothesis of openness can be weakened to some extent in recent theorems of Arhangelskiĭ, Proizvolov and Čoban (see III.2.1, III.2.2 and III.2.3). Using the method of this section we show that in some special cases this can be further weakened. Specifically we obtain the following theorem.

Theorem 3.14. Let f be a continuous finite-to-one mapping of a Hausdorff space X onto a Hausdorff space Y. Suppose that singular points of X and Y do not accumulate and f/S is one-to-one. The following statements are true.

(a) If X and Y are locally compact, then weight $X \leq$ weight Y.

(b) If X is a regular space and if Y is hereditarily paracompact, then X is hereditarily paracompact.

(c) If Y is hereditarily metacompact (respectively, hereditarily Lindelöf), then X is hereditarily metacompact (respectively, hereditarily Lindelöf).

Proof. Let $\bar{X} = W-T$ and let $\bar{f} = r/\bar{X}$. Since f is continuous finite-to-one, by Proposition 3.11 the function $\bar{f}$ is an open continuous finite-to-one map of $\bar{X}$ onto Y. Since X and Y are Hausdorff spaces, by Proposition 3.11 $\bar{X}$ is a Hausdorff space.

(a) Since X and Y are locally compact spaces, by Proposition 3.12 the space $\bar{X}$ is locally compact. By Theorem I.4.1 weight $\bar{X} \leq$ weight Y. Since X is a subspace of $\bar{X}$, weight $X \leq$ weight Y.

(b) Since X and Y are regular spaces, by Proposition 3.13 the space $\overline{X}$ is also regular. By Theorem I.4.6 the space $\overline{X}$ is hereditarily paracompact. Thus X is hereditarily paracompact.

(c) It is immediate in view of Theorem I.4.5 and the fact that X is a subspace of $\overline{X}$.

Note

1. A paper entitled 'On Open extensions of maps' was presented in the Annual meeting of 'Bharat Ganit Parishad' held at Lucknow in April 1969.

2. A paper entitled 'Some results on open extensions of maps' was presented in the 'Kanpur Conference On Functional Analysis and its applications' held at Indian Institute of Technology Kanpur in December 1969.

3. An abstract entitled 'On Open extensions of maps' co-authored with Professor S.P. Franklin has appeared in the 'Notices Ame. Math. Soc., 72T - G95 (1970)'.

4. An abstract entitled 'On Open extensions of maps II' has appeared in the 'Notices Ame. Math. Soc., 72T - G96 (1970)'.

5. A paper entitled 'On Open extensions of maps' co-authored with Professor S.P. Franklin has been accepted for publication in the 'Canadian Journal of Mathematics'.

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